# MATH 420

# COMPLEX VARIABLES

## SESSION no. 3

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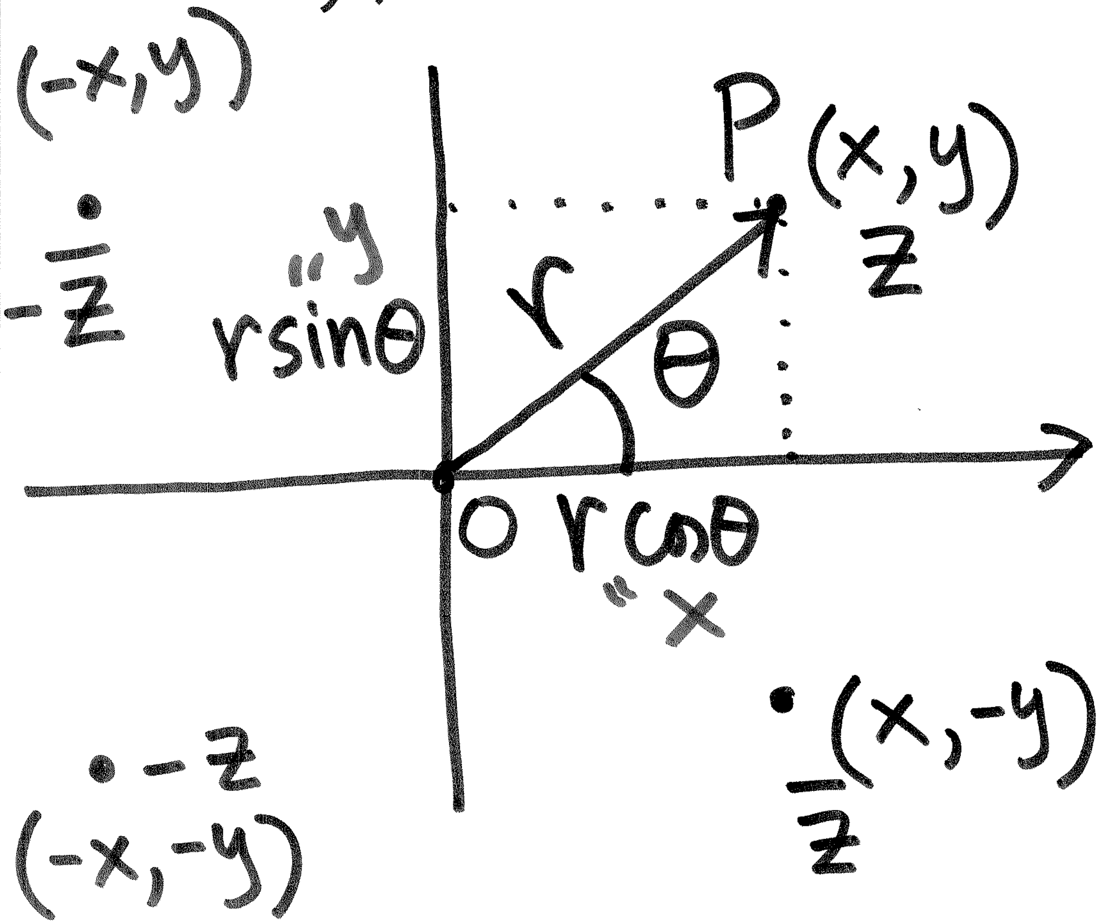


$$z = x + iy$$

$$\bar{z} = x - iy$$

$$-z = -x - iy$$

$$-\bar{z} = -x + iy$$

$$|z| = \sqrt{x^2 + y^2} = |\bar{z}| = |-z|$$

$$r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$$

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$$|z|^2 = z\bar{z}$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

(TRIANGLE INEQ.)

$$|z_1 z_2| = |z_1|\,|z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

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# POLAR REP.

$$z = r(\cos\theta + i\sin\theta)$$

$$0 \leq \theta < 2\pi, \quad r = \sqrt{x^2+y^2}$$

if $z = x + iy$

$$\theta = \tan^{-1} y/x$$

$\hookrightarrow$ argument of $z$

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Polar rep of $z = x + iy$ is

$$z = r(\cos\theta + i\sin\theta)$$
$$= r(\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi))$$

$n$ : any integer

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$0 \leq \theta < 2\pi$ is the principal argument and denoted by $Arg(z)$

Other values of $\theta$ that still satisfy $z = r(\cos\theta + i\sin\theta)$ are denoted by $arg(z)$.

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Ex. Find the polar rep. of a) $z = 3 + i4$ b) $-3 + i4$ c) $-3 - i4$ d) $3 - i4$

$$z = r(\cos\theta + i\sin\theta)$$

$$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

in each case.

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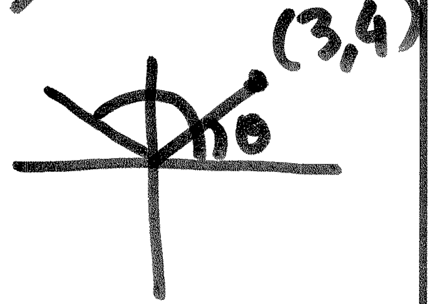
a) $3 + i4 \sim (3,4)$ is in 1st quadrant.




$\theta = \tan^{-1} 4/3$

b) $-3 + i4 \sim (-3,4)$ is in 2nd quadrant

$\theta = \pi - \tan^{-1} 4/3$

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c) ~~$\Theta$~~ $-3 - i4 \sim (-3, -4)$

is in the 3rd quadrant

$$\theta = \pi + \tan^{-1} 4/3$$

d) $3 - i4 \sim (3, -4)$

is in the 4th quad

$$\theta = 2\pi - \tan^{-1} 4/3$$

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Ex. Write $-\sqrt{3}-i$ in polar form.

$r = \sqrt{3+1} = 2$

$-\sqrt{3}-i \sim (-\sqrt{3}, -1) \in$ 3rd quad

$\theta = \pi + \tan^{-1}\frac{1}{\sqrt{3}} = \pi + \frac{\pi}{6}$

$= 7\pi/6$

$-\sqrt{3} - i = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

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# Power of a complex no.

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

CHECK

$$= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

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$$\operatorname*{Arg}_{(a)} z_1 z_2 = \operatorname*{Arg}_{(a)} z_1 + \operatorname*{Arg}_{(a)} z_2$$

$$|z_1 z_2| = |z_1||z_2|$$

Let $z_k = r_k(\cos\theta_k + i\sin\theta_k)$

$k = 1, 2, \cdots, n$

$$z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n \left(\cos(\theta_1 + \cdots + \theta_n) + i\sin(\theta_1 + \cdots + \theta_n)\right)$$

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If $z = r(\cos\theta + i\sin\theta)$

Then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Also

$$z^n = r^n(\cos\theta + i\sin\theta)^n$$

$$\Rightarrow \boxed{(\cos n\theta + i\sin n\theta) = (\cos\theta + i\sin\theta)^n}$$

de Moivre's Formula

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**Example** Prove that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

$$= \underline{\cos^3\theta} + i\,3\cos^2\theta\sin\theta + \underline{3\cos\theta\, i^2\sin^2\theta} + i^3\sin^3\theta$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

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$$\cos^2\theta = 1 - \sin^2\theta$$

$$\Rightarrow \sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta$$

$$= \sin\theta\left(3\cos^2\theta - \sin^2\theta\right)$$

$$= \sin\theta\left(3 - 3\sin^2\theta - \sin^2\theta\right)$$

$$= \sin\theta\left(3 - 4\sin^2\theta\right)$$

$$= 3\sin\theta - 4\sin^3\theta$$

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$$z = r(\cos\theta + i\sin\theta)$$

$$z^{-1} \rightarrow \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{r(\cos\theta - i\sin\theta)}{r^2}$$

$$= \frac{1}{r}\left(\cos(-\theta) + i\sin(-\theta)\right)$$

$$= r^{-1}\left(\cos(-\theta) + i\sin(-\theta)\right)$$

$\Rightarrow$ For all integers $n$ (positive or negative)

$$z^k = r^k(\cos k\theta + i\sin k\theta);$$

$$k = \ldots -1, 0, 1, \ldots$$