MATH 420

COMPLEX VARIABLES

SESSION no. 4

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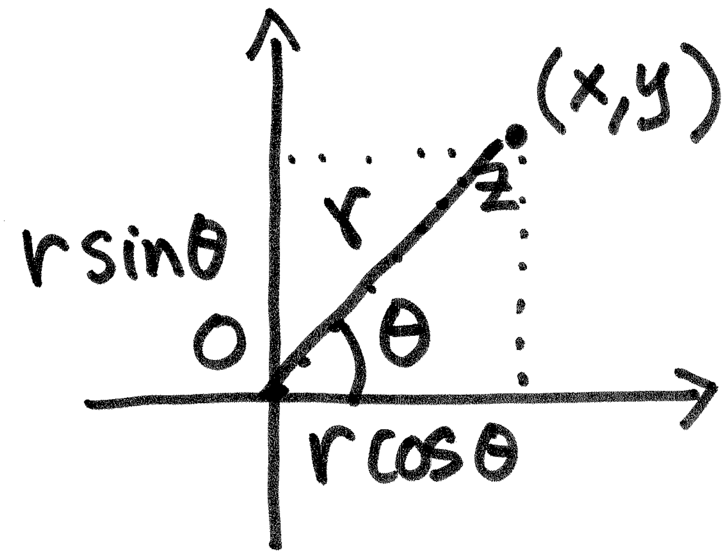
Recall:

$$z = x + iy$$

$$= r(\cos\theta + i\sin\theta)$$

polar form



$$= r(\cos\phi + i\sin\phi)$$

$$\theta = \phi + 2k\pi \; ; \quad k = \pm 1, \pm 2, \ldots$$

If $0 \leq \theta < 2\pi$ the

θ : principal argument

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$\mathrm{Arg}(z)$: $0 \le \theta < 2\pi$

otherwise θ : $\arg(z)$.

For all integers $k = \ldots, -1, 0, 1, \ldots$

$$z^k = r^k(\cos k\theta + i \sin k\theta)$$

where $z = r(\cos\theta + i\sin\theta)$.

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Ex. $z = \sqrt{3} + i$ Find z^7.

$\sqrt{3} + i \sim (\sqrt{3}, 1) \in$ 1st quadrant

$\theta = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$|z| = \sqrt{3+1} = 2$

$z = 2\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$

$Arg(z) = \pi/6$, $arg(z) = \frac{\pi}{6} \pm 2k\pi$

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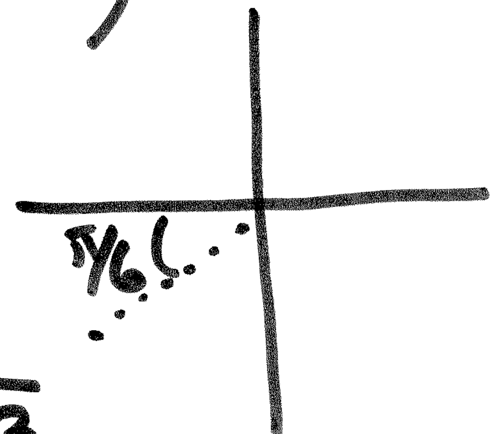
$$z^7 = 2^7\left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6}\right)$$

(polar form of z^7)

$$\frac{7\pi}{6} = \pi + \frac{\pi}{6}$$



$$\cos \frac{7\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$\sin 7\pi/6 = -\sin \pi/6 = -1/2$$

$$z^7 = 2^7\left(-\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right) = -2^6(\sqrt{3}+i)$$

rect. form

⑤

Finding the roots of a complex no.

Given $a \in \mathbb{C}$, find z s.t.

$$z^n = a \quad ; \quad n \geq 2 \text{ ; integer}$$

$$z = a^{1/n}$$

$$a = |a|(\cos\theta + i\sin\theta)$$

$$z = |a|^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

(6)

Another possibity:

$$a = |a|\left(\cos(\theta + 2\pi) + i\sin(\theta + 2\pi)\right)$$

$$z = a^{1/n} = |a|^{1/n}\left(\cos\frac{\theta + 2\pi}{n} + i\sin\frac{\theta + 2\pi}{n}\right)$$

$$a = |a|\left(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)\right)$$

$$z = |a|^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

$$k = 0, 1, \ldots, n-1$$

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$|z| = |a|^{1/n}$

$k=0$: $\arg = \theta/n$

$k=1$: $\arg = \dfrac{\theta+2\pi}{n}$

$\vdots$

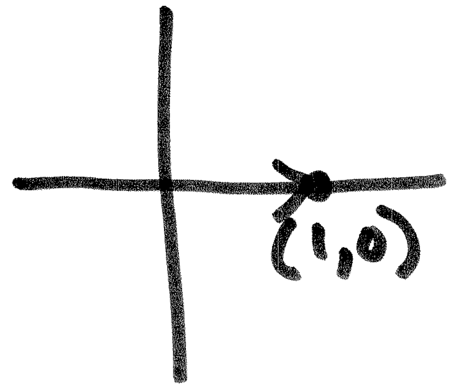
$k=n-1$: $\arg = \dfrac{\theta+2(n-1)\pi}{n}$

$k=n$: $\arg = \dfrac{\theta+2n\pi}{n} = \dfrac{\theta}{n} + 2\pi$

Ex. Roots of unity (1).

Find z s.t. $z = 1^{1/n}$

$$1 = 1(\cos 2k\pi + i\sin 2k\pi)$$



$$k = 0, \pm 1, \pm 2, \ldots$$

nth root of 1:

$$z = 1^{1/n}\left(\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}\right)$$

$$k = 0, 1, \ldots, n-1$$

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$n = 2$; $k = 0, 1$

$k = 0$: $\cos 0 + i \sin 0 = 1$

$k = 1$: $\cos \frac{2\pi}{2} + i \sin \frac{2\pi}{2}$

$= \cos \pi + i \sin \pi = -1$

$n = 3$; $k = 0, 1, 2$

$k = 0$: $\cos 0 + i \sin 0 = 1$

(10) $n=3$

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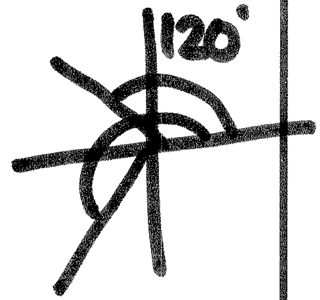
$k=1$: $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$



ω

$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$k=2$: $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

ω^2

$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$\{1, \omega, \omega^2\}$

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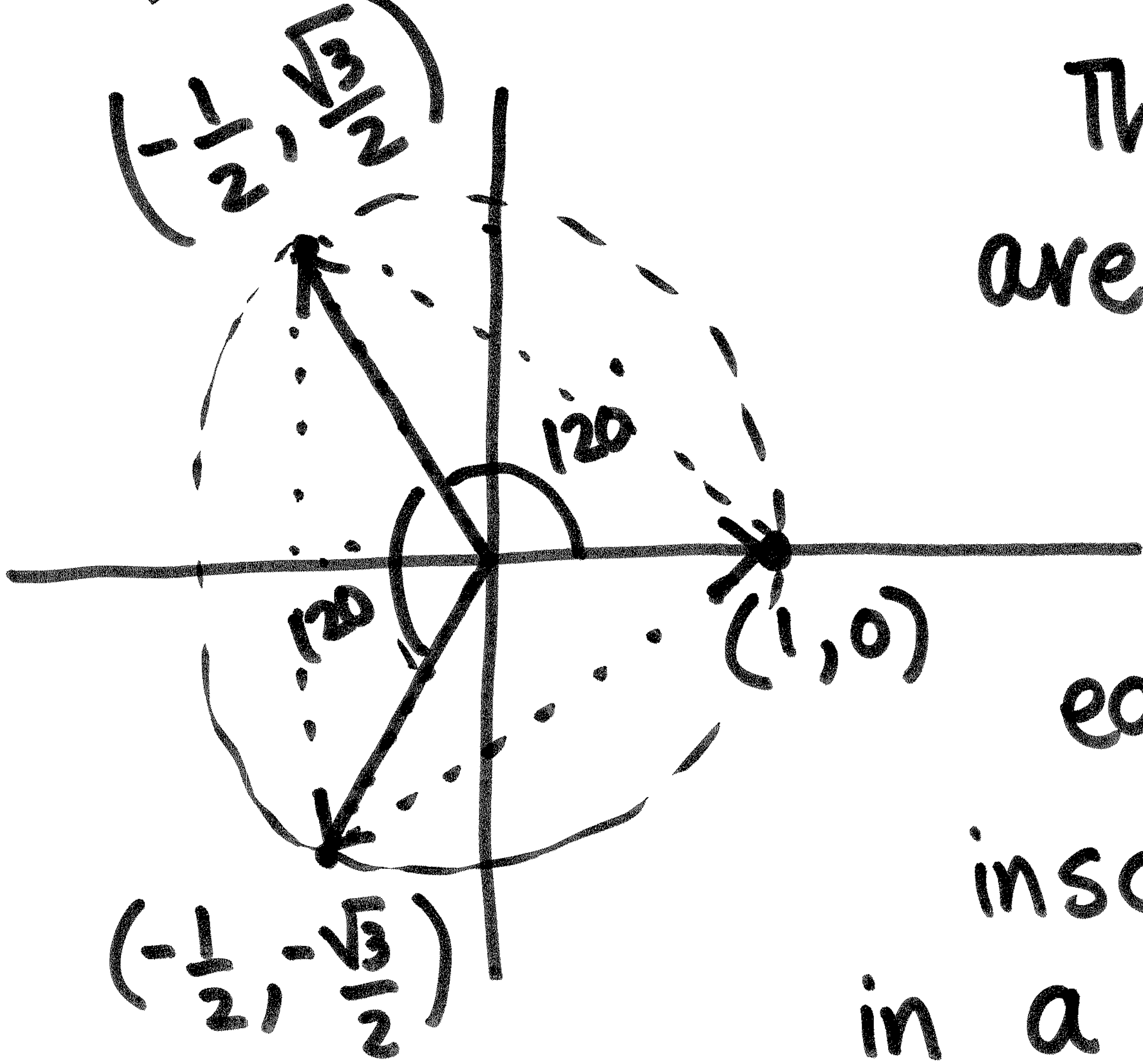
3rd roots of 1



The roots are the vertices of an eq. triangle inscribed in a circle of radius 1.

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In general, the n roots of unity are the vertices of a regular n-polygon inscribed inside a circle of radius 1.