MATH 420

COMPLEX VARIABLES

SESSION no. 5

① Points at infinity

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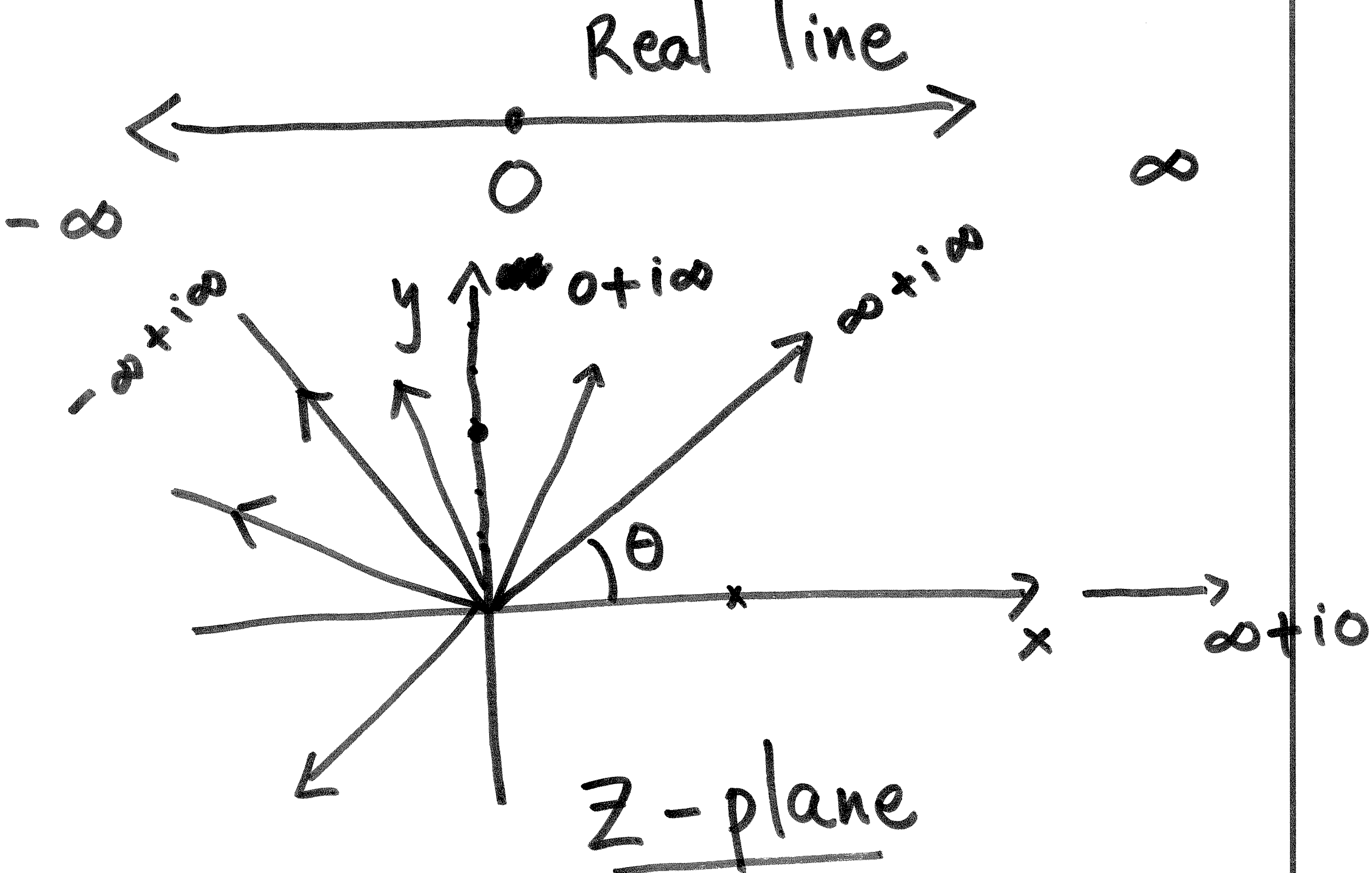
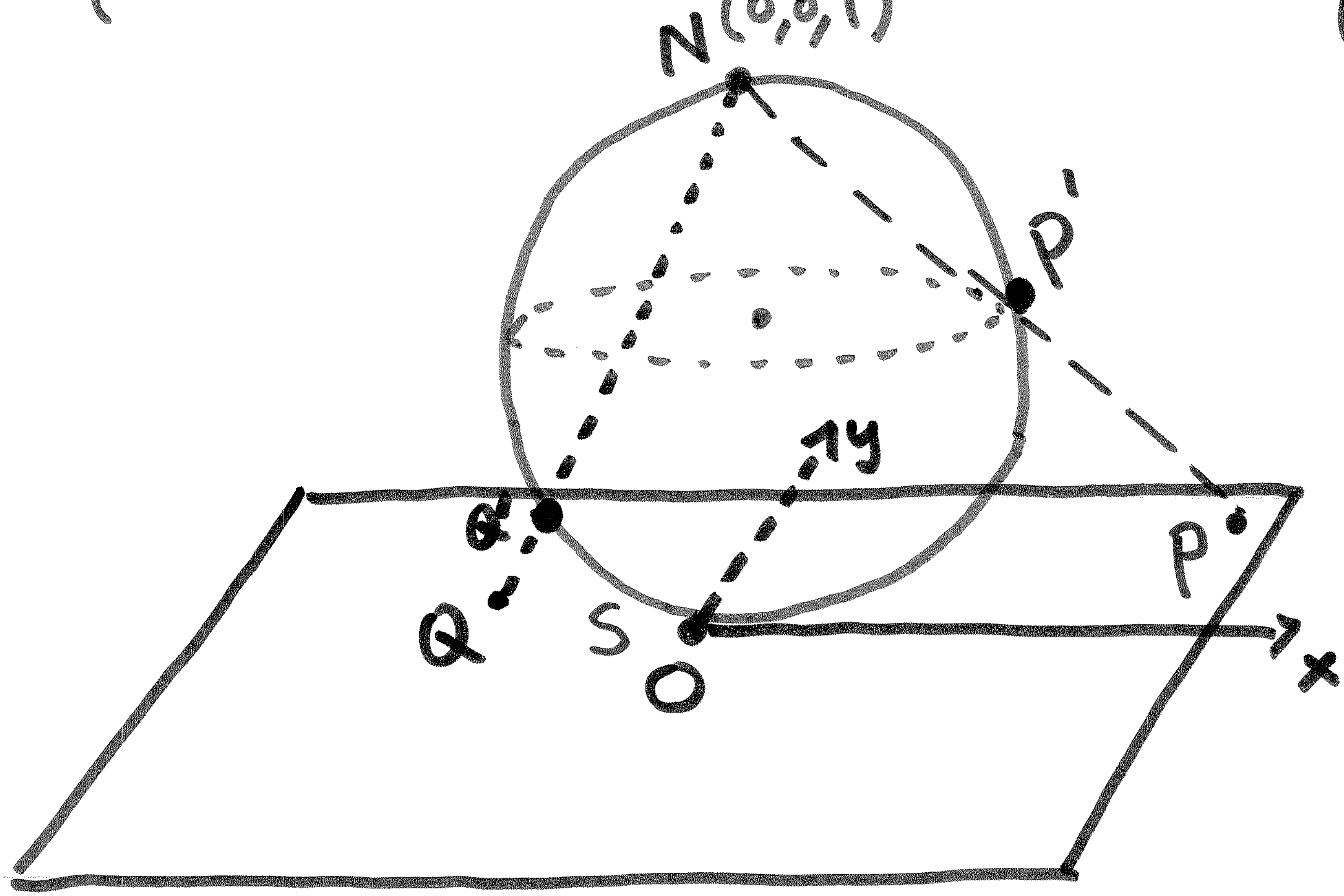


1a)

Stereographic Projection

$S = \{(x,y,z) : x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}\}$; center $(0,0,\frac{1}{2})$



Z - plane

P' – Stereographic projection of P

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- S is tangent to the z-plane at O.

- O stands for the south pole (S)
The point diagonally opposite O – N north pole

- For any P in the z-plane, draw the line joining P to N

- This line strikes the sphere at some P'

- P' – stereographic projection of P.

- All points z s.t. $|z| = \infty$ is mapped to N $(0,0,1)$.

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Functions of a complex variable

$D = \{$set of complex numbers$\}$

A function $\underline{f}$ on D – a rule that assigns to each $\underline{z} \in D$ a complex no. $\underline{w}$.

w – image of z under f

or

value of f at z

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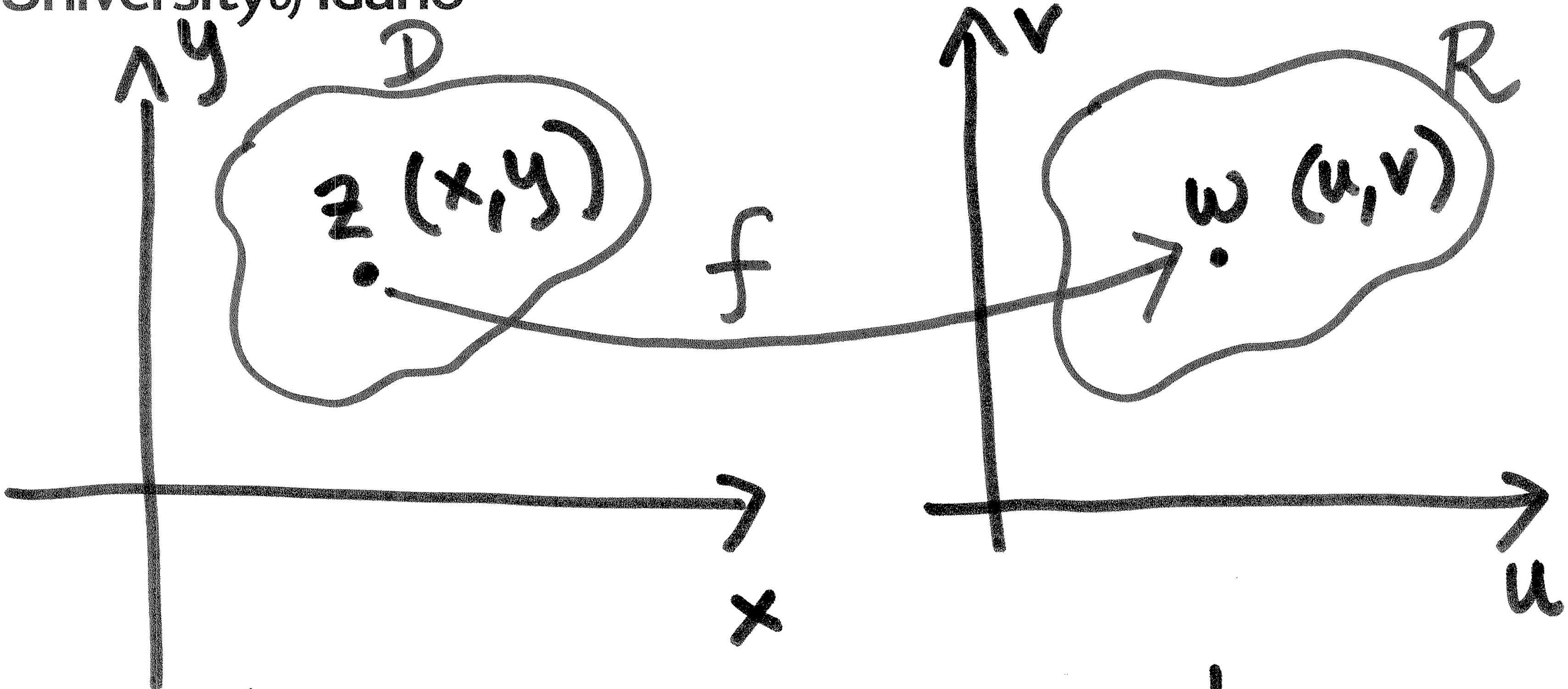

z - plane

$z = x + iy$

w - plane

$w = u + iv$

R - range of f

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Example Let $f(z) = z^2$

$z = x + iy$, Domain $= \mathbb{C}$

$z^2 = (x+iy)(x+iy)$

$= x^2 - y^2 + i\,2xy$

$= \omega = u + iv$

$u = x^2 - y^2$, $v = 2xy$

$u = u(x,y)$, $v = v(x,y)$

$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$

⑥ $z = x + iy$

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Example: $f(z) = \dfrac{1}{z}$

Domain of $f = \mathbb{C} \setminus \{0\}$.

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$$

$$= \frac{x-iy}{x^2+y^2} = f(z) = \omega$$

If $\omega = u + iv$, $u = \dfrac{x}{x^2+y^2}$; $v = \dfrac{-y}{x^2+y^2}$

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$$z = x + iy$$

Ex. $f(z) = |z|^2$

$f(z) = x^2 + y^2$ is real valued.

A function of a complex variable can be real-valued

$f(x) = \sqrt{x}$

$f(4) = \sqrt{4} = +2, -2$

<u>Single-valued</u> & <u>multi-valued functions</u>

<u>Single-valued</u>: One ~~on~~ value of w corresponds to each z under f. Ex: $f(z) = z^2$

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Multi-valued : more than one value of w corresponds to each z under f.

Ex : $f(z) = z^{1/2}$

$$z = r\left[\cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)\right]$$

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$$Z^{1/2} = r^{1/2}\left[\cos\left(\frac{\theta + 2k\pi}{2}\right) + i \sin\left(\frac{\theta+2k\pi}{2}\right)\right]$$

$k = 0, 1$

(1) $k=0 : \quad r^{1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

$k=1 : \quad r^{1/2}\left(\cos\left(\frac{\theta}{2}+\pi\right) + i\sin\left(\frac{\theta}{2}+\pi\right)\right]$

Let $0 \leq \theta < 2\pi$

$$k=1: \quad r^{1/2}\left[-\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right]$$

$$= -\sqrt{r}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]$$

$$\begin{tikzcd}
 & \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \\
z \arrow[ur, bend left, "f"] \arrow[dr, bend right, "f"'] & \\
 & -\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)
\end{tikzcd}$$

many-valued (2-valued) function)