MATH 420

COMPLEX VARIABLES

SESSION no. 6

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Functions of a complex variable

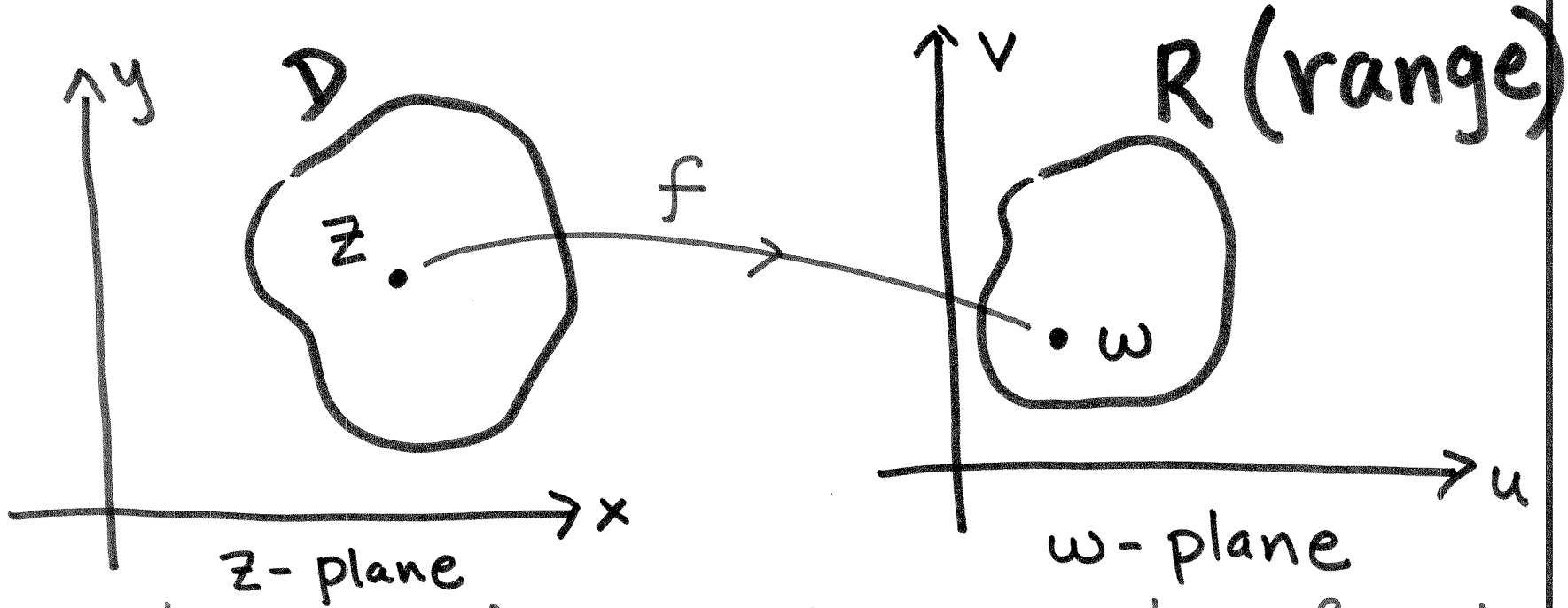
$f : \mathbb{D} \longrightarrow \mathbb{C}$; $\mathbb{D}$ = a set of complex nos.

$z \longmapsto w$

$z = x + iy$, $w = u + iv$

x, y - ind. vars

u, v - dep. vars



f : also called a mapping or transformation

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Single valued functions: each z has only one image under f

Example: $f(z) = z^2, |z|^2$

Multivalued function: each z can have more than one image
(Many-valued)

Example: $f(z) = z^{1/2}$

$z = r(\cos\theta + i\sin\theta)$; $f(z) = \pm\sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

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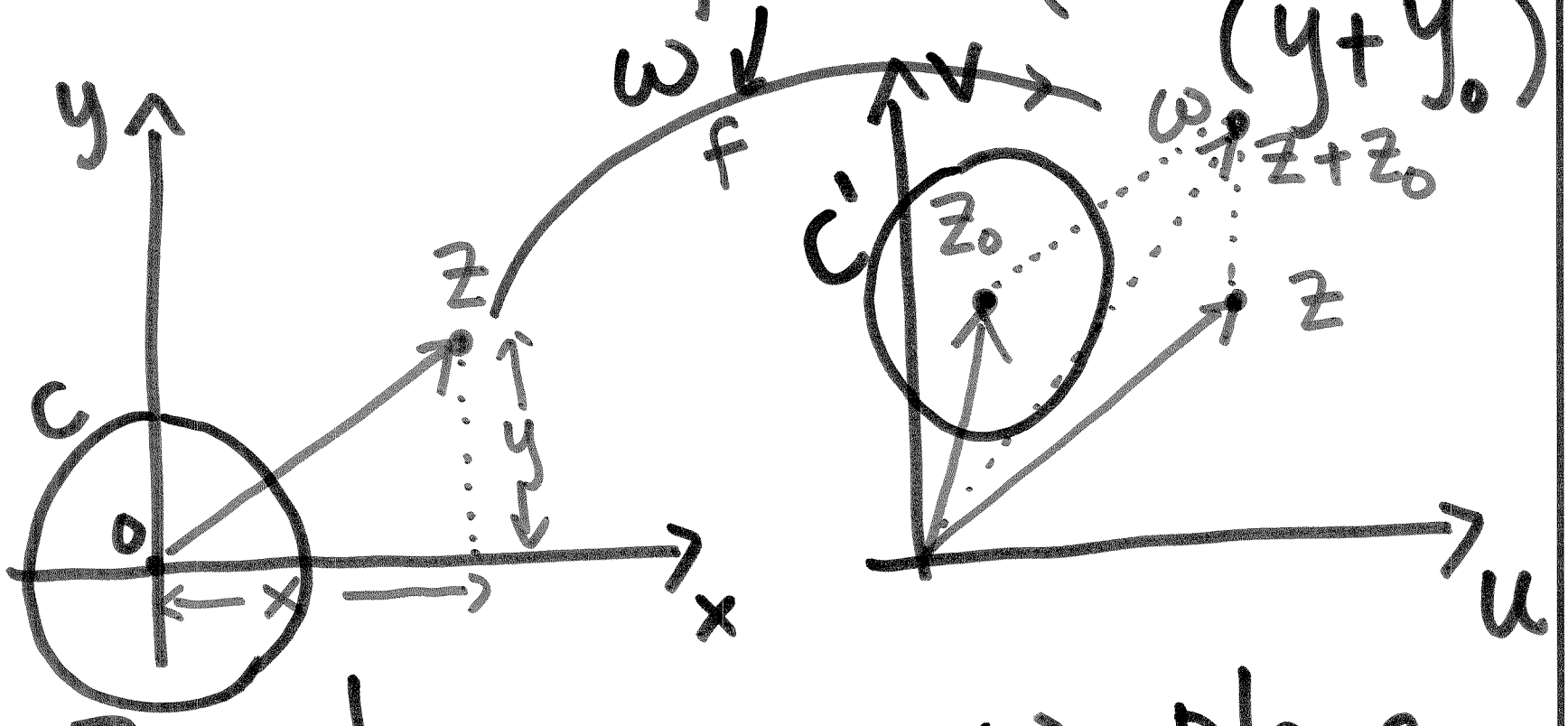
Translation

$z = x + iy$; $f(z) = z + z_0$

$z_0 = x_0 + iy_0$; $f(z) = (x + x_0) + i(y + y_0)$



z - plane

w - plane

$$f(0) = f(0+i0) = 0 + x_0 + i(0+y_0)$$
$$= x_0 + i y_0$$

$f(c) = C'$

Center of $C' = z_0$

radius of C = radius of C'

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Rotation

Example: $\omega = f(z) = iz$

$z = r(\cos\theta + i\sin\theta)$

$i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$f(z)$

$\omega = iz = r\left(\cos\left(\theta+\frac{\pi}{2}\right)+i\sin\left(\theta+\frac{\pi}{2}\right)\right)$

Rotation by $\frac{\pi}{2}$; counterclockwise

$|\omega| = |iz| = |z|$, $\arg(\omega) = \arg(z) + \pi/2$

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Reflection

$$f(z) = \overline{z}$$

A reflection about the real axis.

Inversion: $f(z) = \frac{1}{z}$

$$z = r(\cos\theta + i\sin\theta)$$

$$\frac{1}{z} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$$

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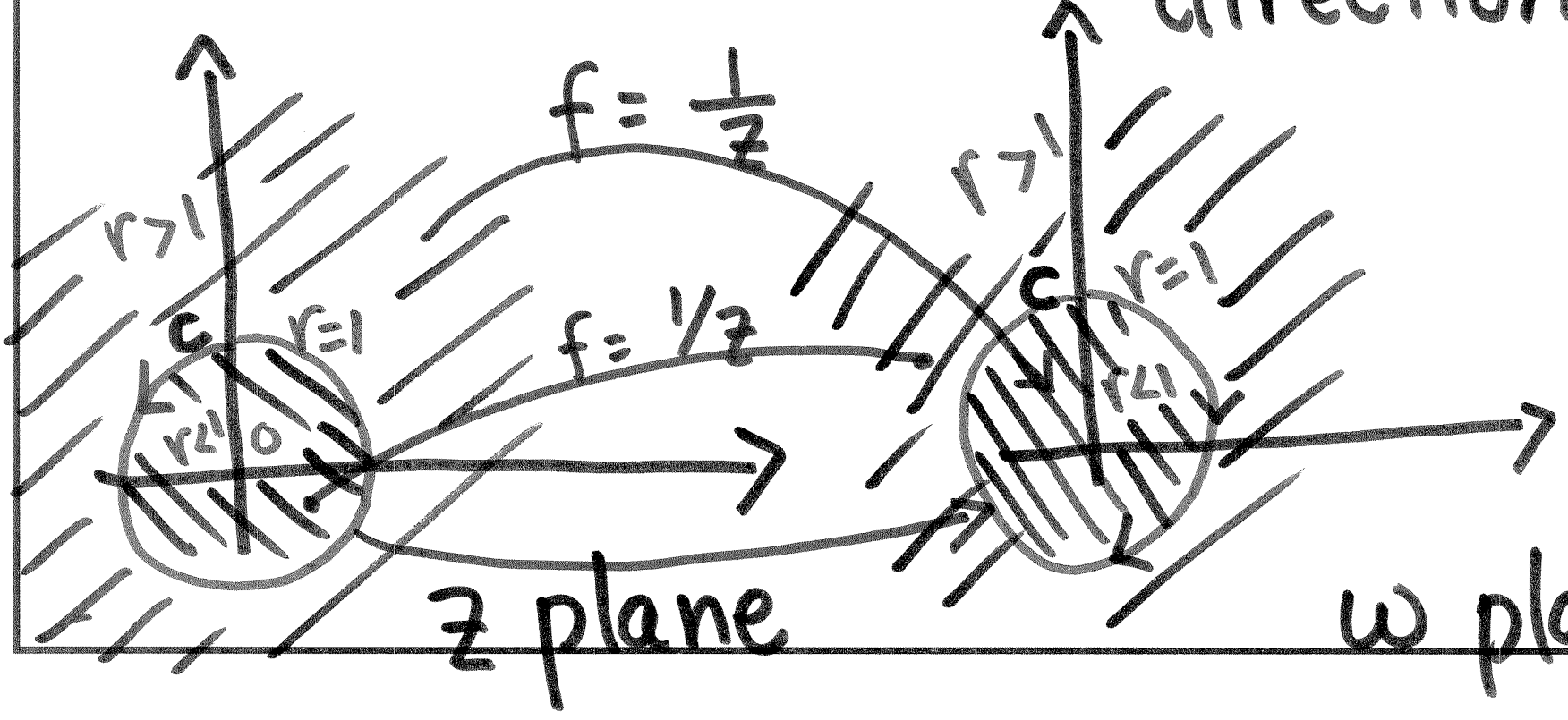
$$w = f(z) = \frac{1}{z} \; ; \; |w| = \frac{1}{r} = \frac{1}{|z|}$$

$$\arg(w) = -\arg(z)$$

$\Rightarrow$ rotation in the opposite direction



$f = 1/z$

If $r > 1$, $\frac{1}{r} < 1$

Outside of C $\xrightarrow{f}$ Inside of C

Inside of C $\xrightarrow{f}$ Outside of C

Boundary $\xrightarrow{f}$ Boundary

(move counter-clockwise)

(move clockwise)

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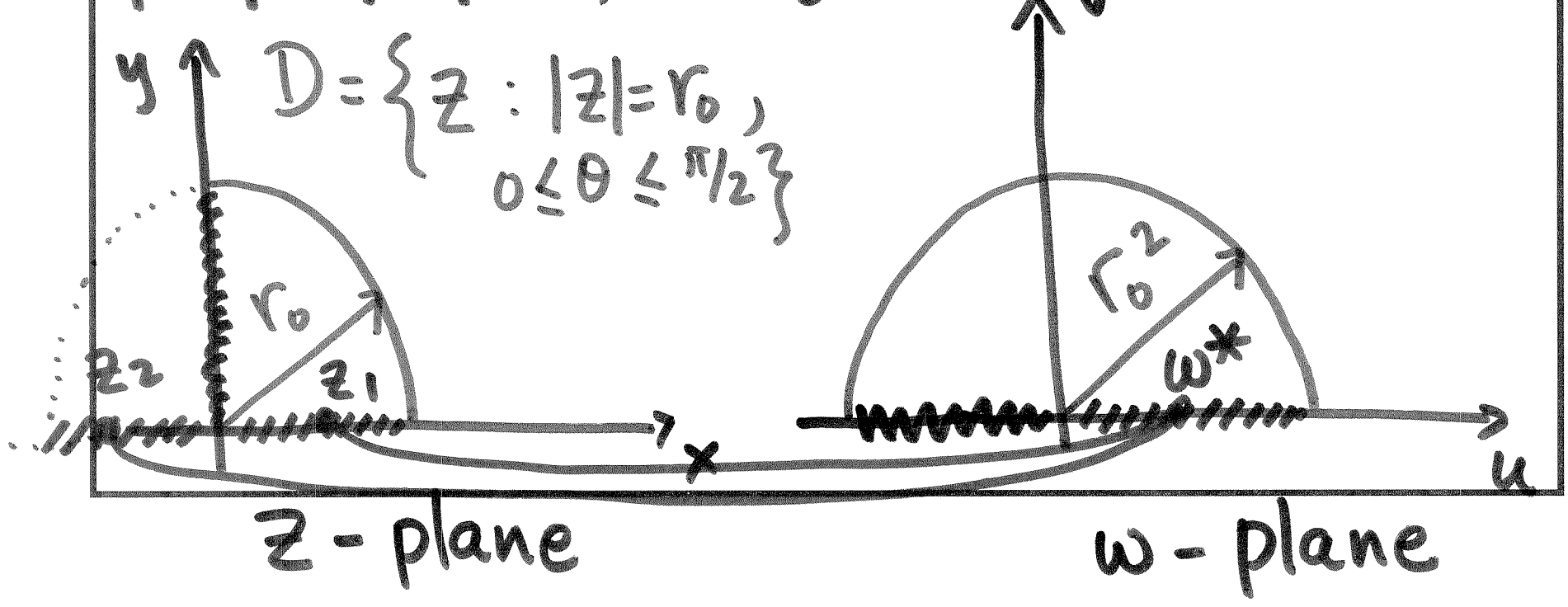
Example $w = z^2 = f(z)$

$$z = r(\cos\theta + i\sin\theta)$$

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

$$|w| = |z|^2 \; ; \; \arg(w) = 2\arg(z)$$



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If D - is in the first quadrant then f is one-to-one map.
- image is in the upper half plane.

Let $D = \{z : |z| = r_0, 0 \le \theta \le \pi\}$

Both the positive and negative real axis in the z-plane

$\xrightarrow{f}$ to the positive real axis in the w-plane

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$f(z) = z^2 \rightarrow$ 2-to-1 map

The upper half plane $\xrightarrow{f = z^2}$ the entire w-plane

$$z_1 = r(\cos\theta + i\sin\theta) \longrightarrow w^* = r^2(\cos 2\theta + i\sin 2\theta)$$

$$z_2 = r(\cos(\theta+\pi) + i\sin(\theta+\pi)) \longrightarrow r^2(\cos(2\theta+2\pi) + i\sin(2\theta+2\pi))$$

$$= r^2(\cos 2\theta + i\sin 2\theta) = w^*$$

same

$z_1, z_2 \xrightarrow{f=z^2} w^*$