MATH 420

COMPLEX VARIABLES

SESSION no. 8

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Continuity

A function f is continuous at z_0 if

a) $\lim_{z \to z_0} f(z)$ exists

b) $f(z_0)$ exists

c) $\lim_{z \to z_0} f(z) = f(z_0)$

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Ex. $$f(z) = \frac{1}{z-1}$$

$f(1)$ does not exist.

f is not continuous at $z = 1$.

Ex.

(fixed) given

$$f(z) = \begin{cases} z^2 & \text{if } z \neq z_0 \\ 0 & \text{if } z = z_0 \end{cases}$$

$$\lim_{z \to z_0} f(z) = z_0^2 \neq f(z_0) = 0$$

unless $z_0 = 0$.

f is discontinuous at $z = z_0 \neq 0$

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Complex Differentiation.

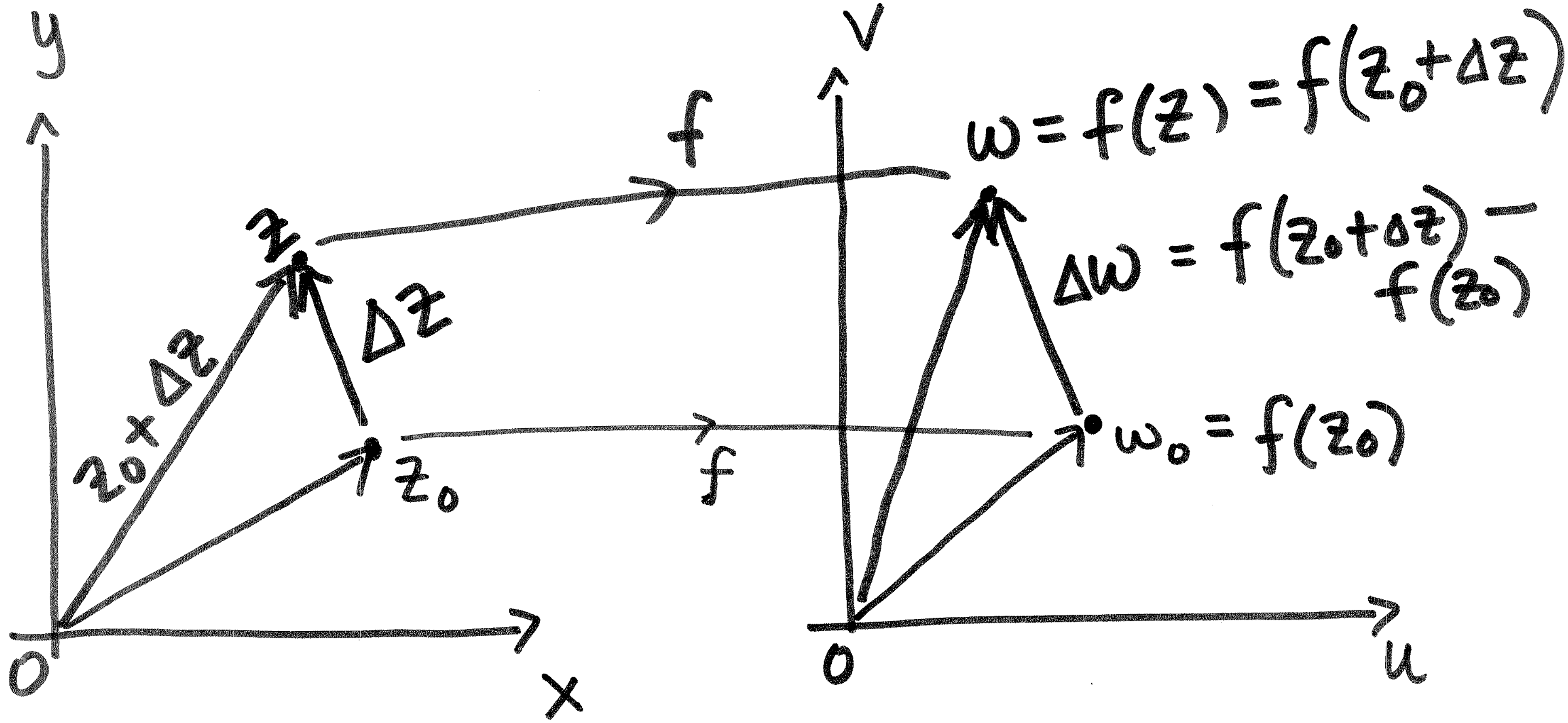
The derivative of f at z_0 :

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) = \frac{df}{dz}$$

Let $\Delta z = z - z_0$ then

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = f'(z_0)$$

Geometrically:



z-plane

w-plane

$w = f(z)$

Ex. $f(z) = z^2$ At any z_0

$$f'(z_0) = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} \frac{(z-z_0)(z+z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} z + z_0 = 2z_0$$

6a

Derivatives of elementary functions

$$\frac{d}{dz} c = 0$$

$$\frac{d}{dz} z^{n} = n z^{n-1} , \quad n \geq 1$$

$$\frac{d}{dz} e^{z} = e^{z}$$

$$\frac{d}{dz} \cos z = -\sin z , \quad \frac{d}{dz} \sin z = \cos z$$

$$\ldots$$

6b

Rules for differentiation

$$\frac{d}{dz}\left[f(z) \pm g(z)\right] = \frac{d}{dz}f \pm \frac{d}{dz}g$$

$$\frac{d}{dz}\left(f(z)g(z)\right) = f'(z)g(z) + f(z)g'(z)$$

$\downarrow$

Product Rule

Ex. $f(z) = \bar{z}$ $\quad$ $z = x + iy$

$z_0 = x_0 + iy_0$

$$f'(z_0) = \lim_{z \to z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

From previous lecture this limit <u>does not exist</u> — f is not differentiable at z_0

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<u>Analytic</u> <u>functions</u> : f' exists at all z is some region R. Then f is said to be analytic in R.

f' or $\dfrac{df}{dz}$ – 1st derivative

f'' or $\dfrac{d^2f}{dz^2}$ – 2nd derivative

$\vdots$

$f^{(n)}$ or $\dfrac{d^nf}{dz^n}$ – nth order derivative

Suppose that $f(z)$ is <u>analytic</u> in a region R. Then so are $f'(z)$, $f''(z)$, $\cdots$, i.e. all higher order derivatives exist in R.

[To be Proved later].

Cauchy - Riemann Eqs.

__Goal__: To find some simple test to determine analyticity of $f(z)$.

Let $f: D \longrightarrow \mathbb{C}$ be analytic

$z = x + iy$

$$
\begin{aligned}
f(z) &= f(x+iy) \\
&= u(x,y) + i\,v(x,y)
\end{aligned}
$$

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$$f = u + iv$$

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$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z)$$

Evaluate this in 2 ways:

① Let $\Delta z \to 0$ through real values

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{f(x+\Delta z+iy) - f(x+iy)}{\Delta z}$$

$$= \frac{u(x+\Delta z, y) - u(x,y)}{\Delta z} + i\,\frac{v(x+\Delta z, y) - v(x,y)}{\Delta z}$$

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As $\Delta z \to 0$ through reals

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} \qquad (1)$$

2 Let $\Delta z \to 0$ through imaginary values

$$\frac{f(z+i\Delta z)-f(z)}{i\Delta z}$$

$$= \frac{f(x+i(y+\Delta z)) - f(x+iy)}{i\Delta z}$$

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$$= i\,\frac{u(x, y+\Delta z) - u(x,y)}{i \cdot i\,\Delta z} +$$

$$\cancel{i}\,\frac{v(x, y+\Delta z) - v(x,y)}{\cancel{i}\,\Delta z}$$

$$= \frac{v(x, y+\Delta z) - v(x,y)}{\Delta z} - i\,\frac{u(x, y+\Delta z) - u(x,y)}{\Delta z}$$

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As $\Delta z \to 0$ through img. values

$$\boxed{f'(z) = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}} \quad (2)$$

Equate (1) & (2)

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

↳ CAUCHY-RIEMANN EQS.