MATH 420

COMPLEX VARIABLES

SESSION no. 9

Cauchy-Riemann Equations

$$f(z) = u(x,y) + i\,v(x,y) \; ; \; z = x + iy$$

$f'(z_0)$ exists $(z_0 = x_0 + iy_0) \Longrightarrow$

C-R eqs derived last class

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0)$$

$$\frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0)$$

$z = x + iy$

Also,

$$\boxed{f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$$

or,

$$\boxed{f'(z_0) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$$

Ex: $f(z) = z^2$ is analytic

$$= (x+iy)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + i\underbrace{2xy}_{v(x,y)}$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial v}{\partial y} = 2x$$

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$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \quad ✓$$

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} \quad ✓$$

The C-R eqs hold - as expected.

~~If f is not analytic at z_0~~

~~$\Rightarrow$~~

$Z = x + iy$

If the C-R eqs. fail at z_0

$\Longrightarrow$ f is not analytic at z_0

Ex: $f(z) = |z|^2 = \underbrace{x^2 + y^2}_{u(x,y)} + i\underbrace{0}_{v(x,y)}$

$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$

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$f(z) = |z|^2$

$$\frac{\partial u}{\partial x} = 2x \neq \frac{\partial v}{\partial y} = 0 \quad \text{unless } x = 0$$

$$\frac{\partial u}{\partial y} = 2y \neq -\frac{\partial v}{\partial x} = 0 \quad \text{unless } y = 0$$

$f'(z)$ does not exist at $z \neq 0$.

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$z = x + iy$

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C-R eqs. are only necessary; not sufficient.

$$\text{Ex}: f(z) = \begin{cases} \underbrace{\dfrac{x^{4/3}y^{5/3}}{x^2+y^2}}_{u(x,y)} + i\underbrace{\dfrac{x^{5/3}y^{4/3}}{x^2+y^2}}_{v(x,y)} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

At $z=0$: $\dfrac{\partial u(0,0)}{\partial y} = \lim\limits_{h\to 0} \dfrac{u(0,h)-u(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{u(0,h)}{h}$

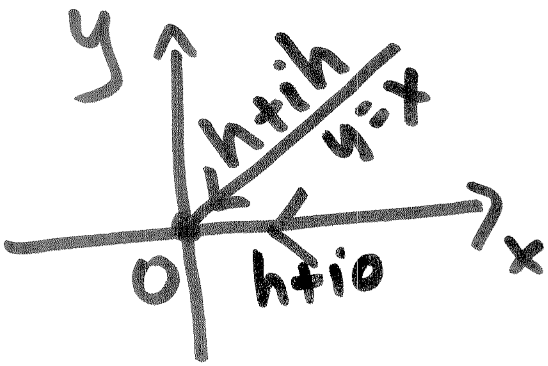
$$\frac{\partial u}{\partial y}(0,0) = \lim_{h \to 0} \frac{0^{4/3} h^{5/3}}{0^2 + h^2} \frac{1}{h} = 0$$

$$\frac{\partial u}{\partial x}(0,0) = 0 = \frac{\partial v}{\partial x}(0,0) = \frac{\partial v}{\partial y}(0,0) = 0$$

$\Rightarrow$ C-R eqs are satisfied at the origin

But $f'(0)$ does not exist :

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z}$$



1) Let $\Delta z \to 0$ along reals

$\Delta z = h \in \mathbb{R}$

$$\lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \frac{1}{h} \frac{h^{4/3} 0^{5/3} + i h^{5/3} 0^{4/3}}{h^2 + 0^2}$$

$$= 0$$

2) Let $\Delta z = h + ih$

$$f'(0) = \lim_{h \to 0} \frac{f(h+ih)}{h+ih} = \lim_{h \to 0} \frac{1}{h+ih} \frac{h^{4/3} h^{5/3} + i h^{4/3} h^{5/3}}{h^2 + h^2}$$

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$$f'(0) = \lim_{h \to 0} \frac{\cancel{(1+i)}\, h^3}{h\,\cancel{(1+i)}\, 2h^2} = \frac{1}{2} \neq 0$$

$f'(0)$ does not exist.

C-R eqs are satisfied at $(0,0)$ but $f'(0)$ does not exist.

Sufficient conditions for analyticity:

$f = u(x,y) + i\,v(x,y)$

a) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ exist

and are <u>continuous</u> at z_0

b) C-R eqs. hold $\left[\begin{array}{l} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{array}\right.$

then $f'(z_0)$ exists