MATH 420

COMPLEX VARIABLES

SESSION no. 11

University of Idaho

Elementary Functions

$$e^{i\theta} = \cos\theta + i\sin\theta \; ; \; \theta \text{ real}$$

Euler's formula

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$z^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$$

before

University of Idaho

$$z = x + iy$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\cos z := 1 - z^2/2! + z^4/4! - \cdots$$

$$\sin z := z - z^3/3! + z^5/5! - \cdots$$

$$\tan z = \frac{\sin z}{\cos z} \; ; \; \sec z = \frac{1}{\cos z} \cdots$$

University of Idaho

$$e^{iz} = \cos z + i \sin z$$

$$e^{-iz} = \cos z - i \sin z$$

$$\frac{e^{iz} + e^{-iz}}{2} = \cos z$$

$$\frac{e^{iz} - e^{-iz}}{2i} = \sin z$$

4

$z = x + iy \,; \; f(z) = e^z$

University of Idaho

$w = e^z = e^{x+iy} = e^x e^{iy}$

Let $w = re^{i\theta}$

$\Rightarrow re^{i\theta} = e^x e^{iy}$

$\Rightarrow \boxed{\begin{aligned} r &= e^x \\ \theta &= y \end{aligned}}$

University of Idaho

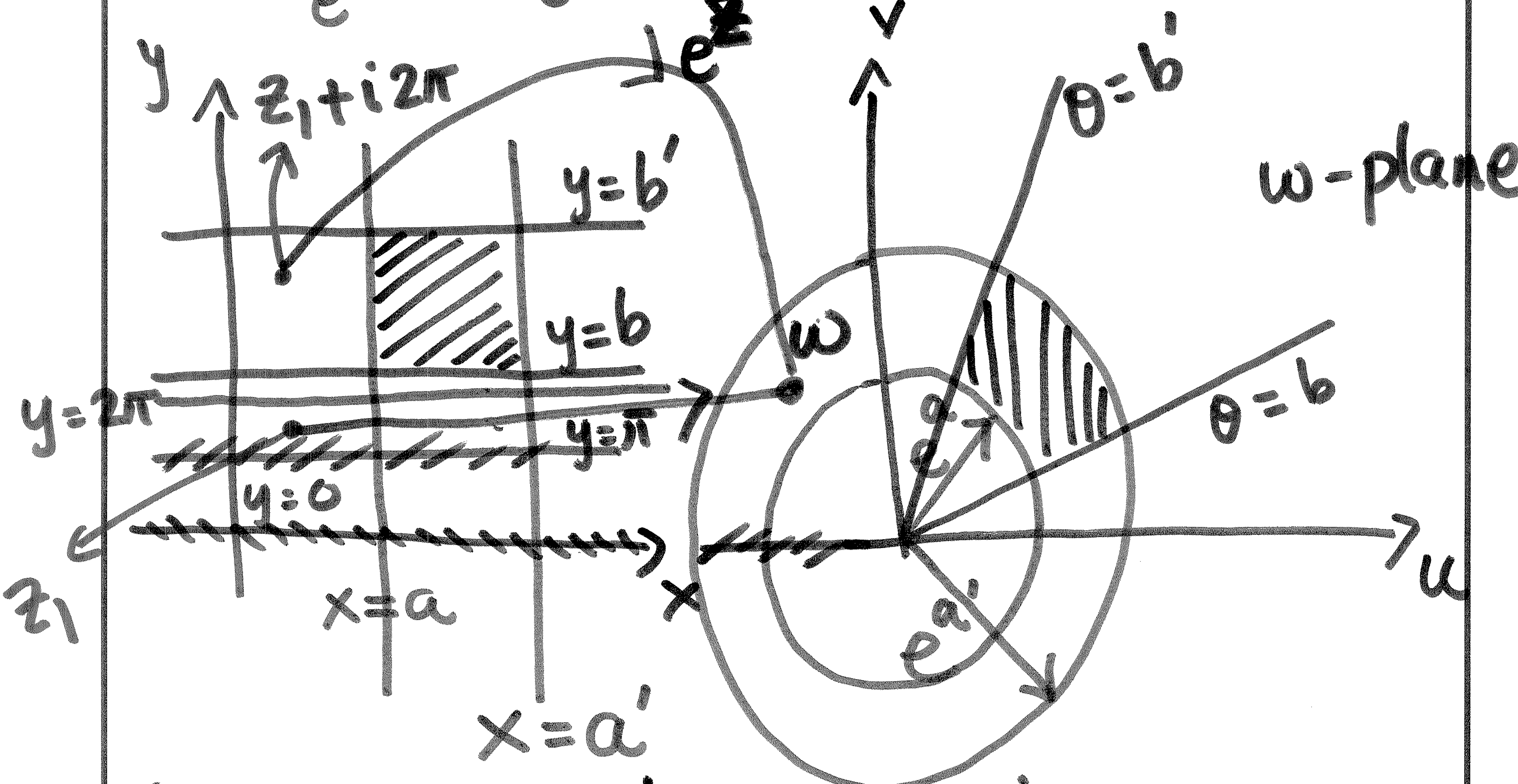
$z = x + iy \,, \quad e^{z} = w = re^{i\theta}$

$e^{x+iy} = re^{i\theta}$

$e^{x}e^{iy} = re^{i\theta}$

$r = e^{x} \,, \quad \theta = y$

$e^{a} \qquad \theta = b$



$y = 0 \longrightarrow$ positive u axis

$y = \pi \longrightarrow$ negative u-axis

$$\{a \le x \le a', b \le y \le b'\} \xrightarrow{e^z}$$

$$\{re^{i\theta} : e^a \le r \le e^{a'}, b \le \theta \le b'\}$$

Strip $0 \le y \le \pi \longrightarrow$ upper half w-plane

$\pi \le y \le 2\pi \longrightarrow$ lower half w-plane

$0 \le y \le 2\pi \longrightarrow$ entire w-plane

7

University of Idaho

∞ly many strips of width $2\pi \xrightarrow{e^z}$ entire w plane

$2n\pi \leq y \leq 2(n+1)\pi$; $n = \cdots -2, -1, 0, 1, 2, \cdots$

$\longrightarrow$ entire w plane

$$w = e^{z_1} = e^{x_1+iy_1} = e^{x_1} e^{iy_1} = e^{x_1} e^{i(y_1+2n\pi)}$$

$$= e^{x_1+iy_1+i2n\pi} = e^{z_1+i2n\pi}$$

7a

University of Idaho

$$\left.\begin{array}{l}\text{Many } z\text{s} \\ z + i2n\pi\end{array}\right\} \xrightarrow{e^z} \text{same } w$$

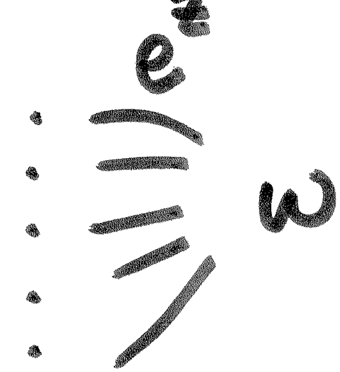
exp is a many-to-one map.

Since $e^{z_1} = e^{z_1 + i2n\pi}$

the exp is periodic with period $i2n\pi$; n is an integer

Logarithm

e^z is many-one



Cannot define log as inverse of e^z.

Want: $w = e^z$ when $z = \log w$

given (w)

find (z)

9

University of Idaho

$$z = x + iy$$

$$w = e^{z} = e^{x+iy} = e^{x} e^{iy}$$

$$|w| = e^{x} \Rightarrow x = \log|w|$$

$$y = \operatorname{Arg} w + 2n\pi \ ; \ n \text{ integer}$$

$$0 \le \operatorname{Arg} w < 2\pi$$

$$\underset{x+iy}{z} = \log w = \{\log|w| + i(\operatorname{Arg} w + 2n\pi)\}$$

10

University of Idaho

$$\operatorname{Log} w = \{\log|w| + i \operatorname{Arg} w\}$$