MATH 420

COMPLEX VARIABLES

SESSION no. 13

Practice Problems for Midterm I

MATH 420, Spring 2012

1. Describe the set determined by the given condition: (a) $|z - 4i| \geq 4$ (b) $|z - 1 + i| = 1$

2. Find
   (a) $\overline{iz}$ (b) the modulus of $-5 - i\sqrt{11}$

3. Write in polar form
   (a) $i(1 - i\sqrt{3})$ (b) $2(1 + i\sqrt{3})$

4. Use de Moivre's Theorem to show that

$$\sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta$$

5. Find $(-1 - i)^4$

6. Find all values of
   (a) $(-8i)^{1/3}$ (b) $(-1)^{1/3}$

7. Find $\lim_{z\to\infty} \frac{1-z}{z^2+1}$
   (Ans = 0)

8. Find $\lim_{z\to i} z^2 + 2z$
   (Ans. $-1 + 2i$)

9. Is the function $f(z) = \frac{1}{z-i}$ continuous at $z = i$? Why, or why not?
   (Ans. Not continuous at $z = i$ because $f(i)$ does not exist)

10. Check the C-R equations for $f(z) = x^3 + i(1 - y)^3$ and find the set where the function is not analytic.
    (Ans. Everywhere except the point $x = 0$, $y = 1$ or, everywhere except the number i.)

11. Show that the function $\log z$ is not continuous along the positive real axis (refer to Lecture 12).

12. Using Euler's formula write $e^{i\pi/2}$ in polar form and find its value.
    (Ans. $e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = i$)

13. Show that $e^{2+3\pi i} = -e^2$

14. Find the logarithm of $-1 - \sqrt{3}i$ and give the principal value.
(Ans. $\log(-1 - \sqrt{3}i) = \log 2 + i(4\pi/3 + 2n\pi)$ and $\mathrm{Log}(-1 - \sqrt{3}i) = \log 2 + i4\pi/3$)

1.

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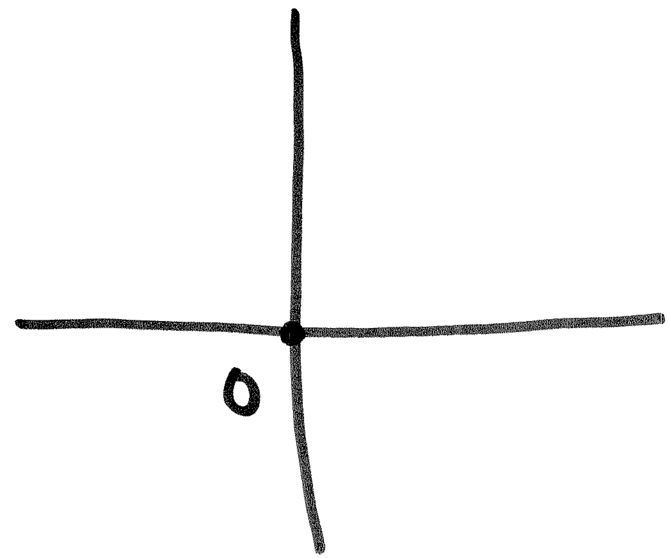
Review

#7 $$\lim_{z \to \infty} \frac{1-z}{z^2+1}$$



$$= \lim_{z \to 0} \frac{1-1/z}{\frac{1}{z^2}+1} \cdot \frac{z^2}{z^2}$$

$$= \lim_{z \to 0} \frac{z^2-z}{1+z^2} = 0$$

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$$\lim_{z \to i} \frac{1}{(z-i)^3}$$

$$\lim_{z \to i} (z-i)^3 = 0 \implies \lim_{z \to i} \frac{1}{(z-i)^3} = \infty$$

$\lim_{z \to z_0} f(z) = \infty$ if and only if $\lim_{z \to z_0} \frac{1}{f(z)} = 0$.

$$\lim_{z \to z_0} f(z) = L$$

1) $z_0 = \infty$

2) $L = \infty$, z_0 is not infinity

3) $z_0 = \infty$, $L = \infty$

$$\lim_{z \to \infty} f(z) = \infty \quad \text{iff} \quad \lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$$

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#8 $\lim_{z \to i} z^2 + 2z$

$= i^2 + 2i$

$= -1 + 2i$

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Cauchy-Riemann Eqs.

#10

$$f(z) = x^3 + i(1-y)^3 = u + iv$$

where is f not analytic?

$$u(x,y) = x^3, \quad v(x,y) = (1-y)^3$$

$$\frac{\partial u}{\partial x} = 3x^2, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial y} = -3(1-y)^2$$

6a

Find points where C-R fail.

C-R eqs.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad 3x^2 = -3(1-y)^2$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad ✓ \quad 0 = -0$$

If C-R eqs hold then

$$3x^2 + 3(1-y)^2 = 0$$

$$x^2 + (1-y)^2 = 0$$ only if

$x = 0$ & $y = 1$.

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C-R equations fail everywhere except $x=0, y=1$ or $z=i$

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$$f(z) = z^2 = \underbrace{x^2 - y^2}_{u} + i\,\underbrace{2xy}_{v}$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = 2y \qquad \frac{\partial v}{\partial y} = 2x$$

② continuous

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}\,; \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

① C-R eqs. are satisfied everywhere

$\Rightarrow$ f is analytic everywhere

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

$$= \partial v/\partial y - i\,\partial u/\partial y$$

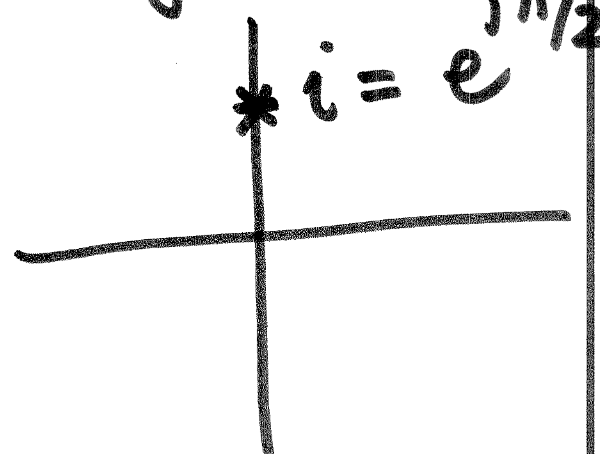
$$f'(z) = 2x + i\,2y = 2(x+iy)$$

$$= 2z$$

#12 $z = 1 \cdot e^{i\pi/2}$ Write in polar form

$$e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \rightarrow \text{polar form}$$

$$= i \longrightarrow \text{rectangular}$$



rectangular

$$Z = x + iy \quad ①$$

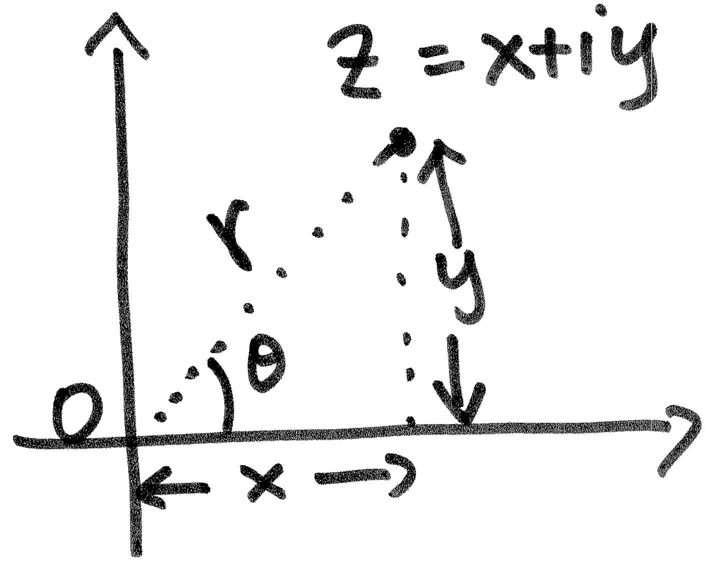
polar form

$$= r(\cos\theta + i\sin\theta) \quad ②$$

$$= re^{i\theta} \quad ③$$ (Euler's formula)

$\mathrm{Re}\, z = x$

$\mathrm{Im}\, z = y$



$|z| = r \rightarrow$ magnitude

$\arg z = \theta$

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\#13 Show $z = e^{2+3\pi i} = -e^2$

$|z|$

$$e^{2+3\pi i} = e^2 e^{3\pi i}$$

$$= e^2 (\cos 3\pi + i \sin 3\pi)$$

$$= e^2 \cos \pi = -e^2$$