MATH 420

COMPLEX VARIABLES

SESSION no. 14

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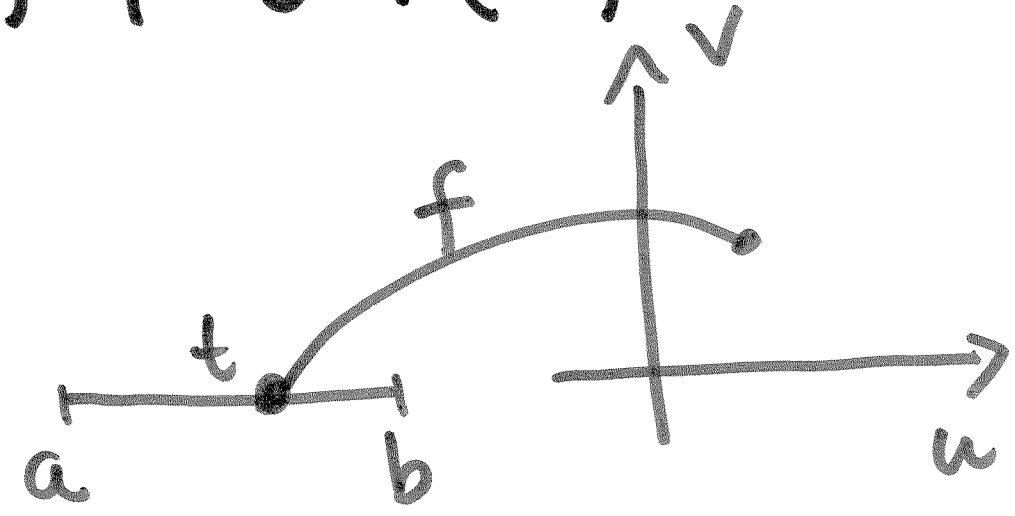
Complex Integration

Integrate a complex func. of a real variable t

t : real variable

$$f(t) = u(t) + i\,v(t)$$

$$a \leq t \leq b$$



Defn. $$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Ex. $f(t) = (1+it)^2$, $0 \le t \le 1$

$$\int_0^1 f(t)\,dt = \int_0^1 (1-t^2)\,dt + i\int_0^1 2t\,dt$$

$$= \left[t - \frac{t^3}{3}\right]_0^1 + i\,t^2\Big|_0^1 = \frac{2}{3} + i$$

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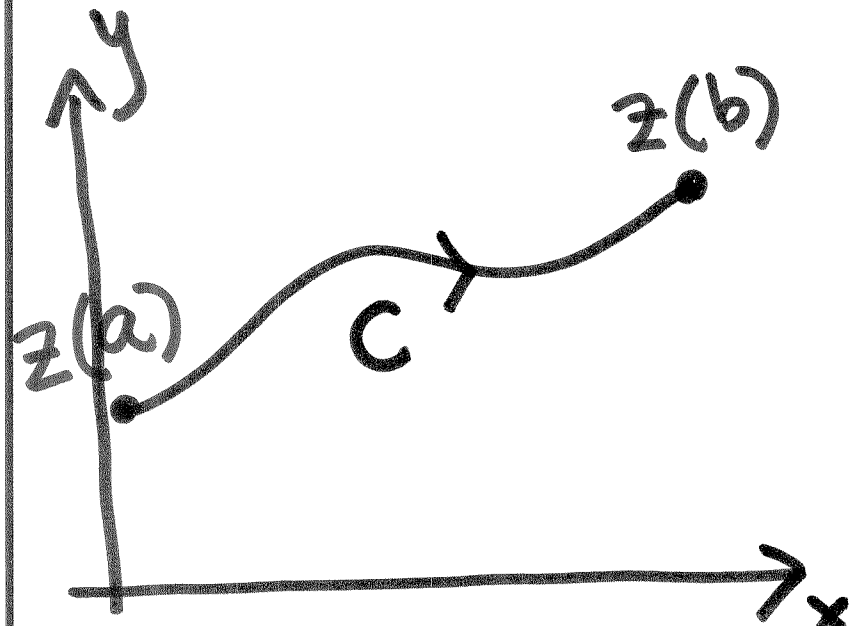
Integrating a complex func. of a complex var.

Curve: $z(t) = x(t) + i\,y(t)$;

$$\boxed{a \le t \le b}$$



Goal:

$$\int_C f(z)\,dz$$

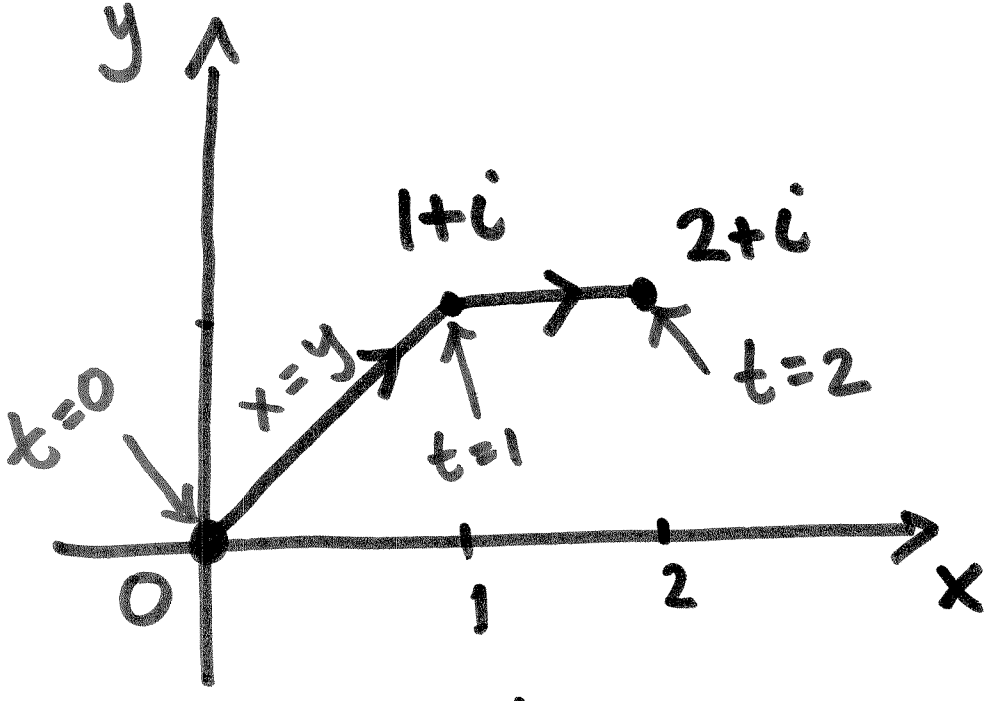
Ex. $z(t) = \begin{cases} t + it & 0 \le t \le 1 \\ t + i & 1 \le t \le 2 \end{cases}$

$x = y$

$t = 0 \to 0 + i0$

$t = 1 \to 1 + i \sim (1,1)$

$t = 2 \to 2 + i$



z - plane

$z'(t) = \begin{cases} 1 + i & 0 \le t \le 1 \\ 1 & 1 \le t \le 2 \end{cases}$

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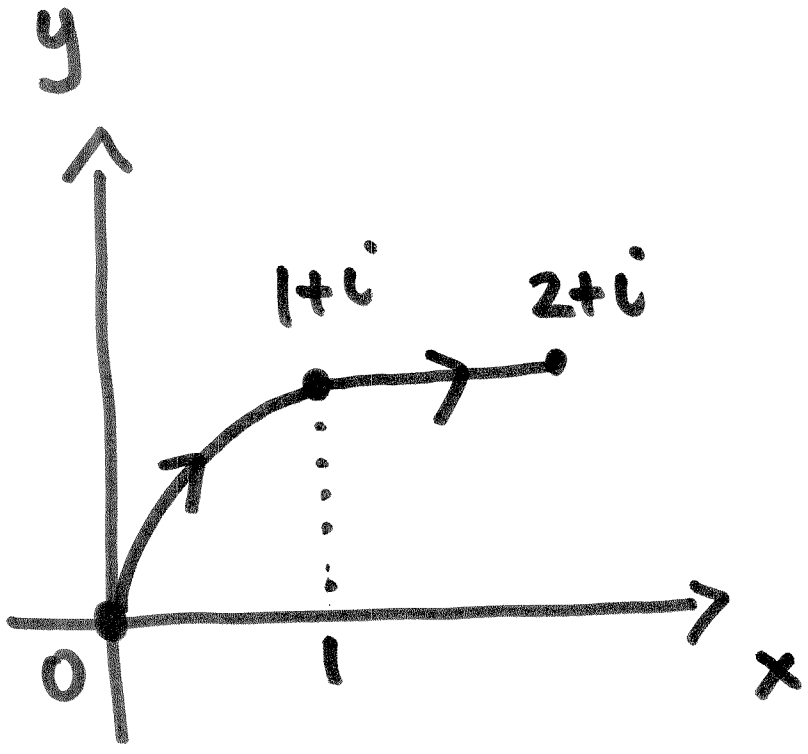
$x = y^2$

Ex. $z(t) = \begin{cases} t^2 + it & 0 \le t \le 1 \\ t + i & 1 \le t \le 2 \end{cases}$

$t = 0 \rightarrow 0 + i0$

$t = 1 \rightarrow 1 + i$

$t = 2 \rightarrow 2 + i$



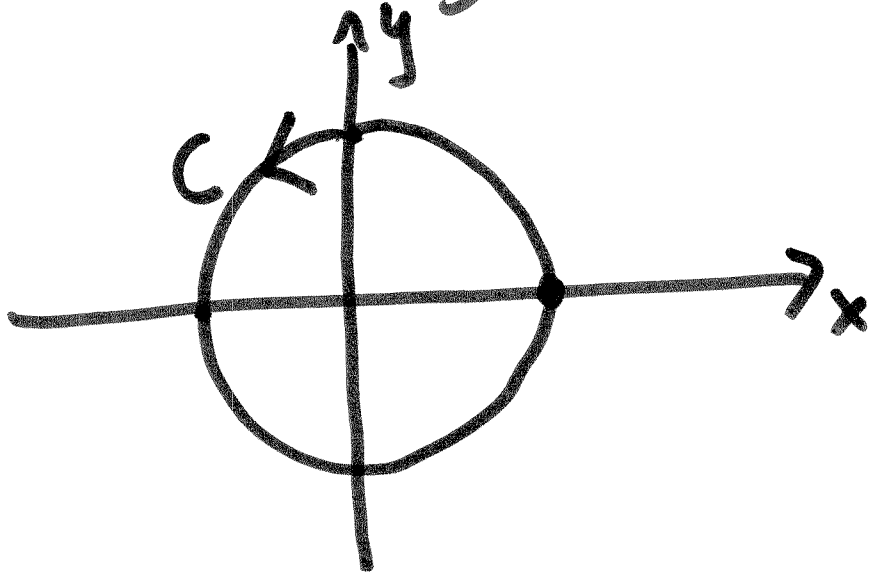
z - plane

e^{i2t}

Ex: $C: z(t) = e^{it}$, $0 \le t \le 2\pi$

$z(t) = \underset{x}{\cos t} + i \underset{y}{\sin t}$

$x^2 + y^2 = 1$



moving counterclockwise along the circle centered at (0,0) and radius = 1.

$z(t) = x(t) + i\,y(t)$, $a \le t \le b$ $\quad [a,b]$

$z'(t) = x'(t) + i\,y'(t)$

If $x'(t)$ & $y'(t)$ are continuous on $[a,b]$ then $z(t)$ is said to be

differentiable.

Smooth curve : If $z(t)$ is differentiable and $z'(t) \neq 0$ for $a < t < b$

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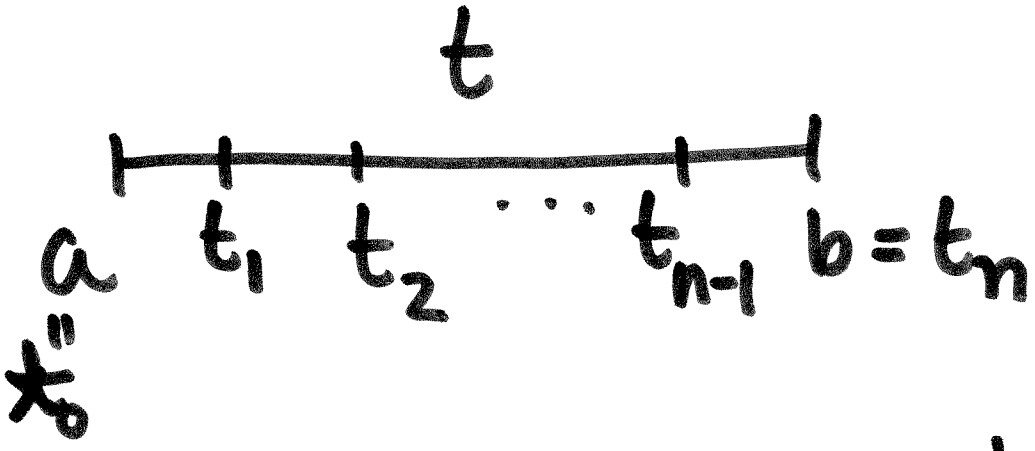
$a \neq b$

Contour:



$$a = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n-1} \leq b = t_n$$

$z(t)$ is a contour or piecewise smooth if $z(t)$ is smooth on each subinterval $[t_{i-1}, t_i]$

$i = 1, \ldots, n$

$z(t) = x(t) + i\,y(t)$

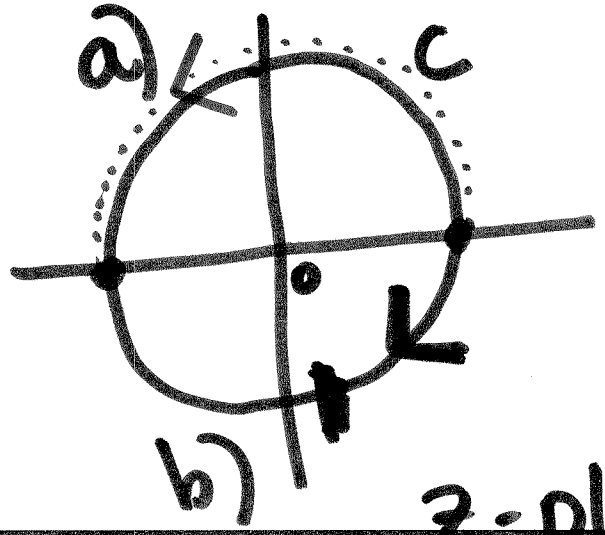
$a \le t \le b$

$$\int_C f(z)\,dz$$

$$= \int_a^b f(z(t))\; z'(t)\,dt$$

Ex. Evaluate $\int_C z^2 |z| \, dz$

a) upper half of the circle $|z| = 2$ (counterclockwise)



$z(t) = 2e^{it}$

$0 \le t \le \pi$

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$dz = 2ie^{it}dt$

$$\int_0^{\pi} 2^2 e^{i2t}\, 2\, 2ie^{it}\, dt = \dots$$

$$= -\frac{32}{3}$$

b) same function but over the lower half circle (going clockwise)