MATH 420

COMPLEX VARIABLES

SESSION no. 15

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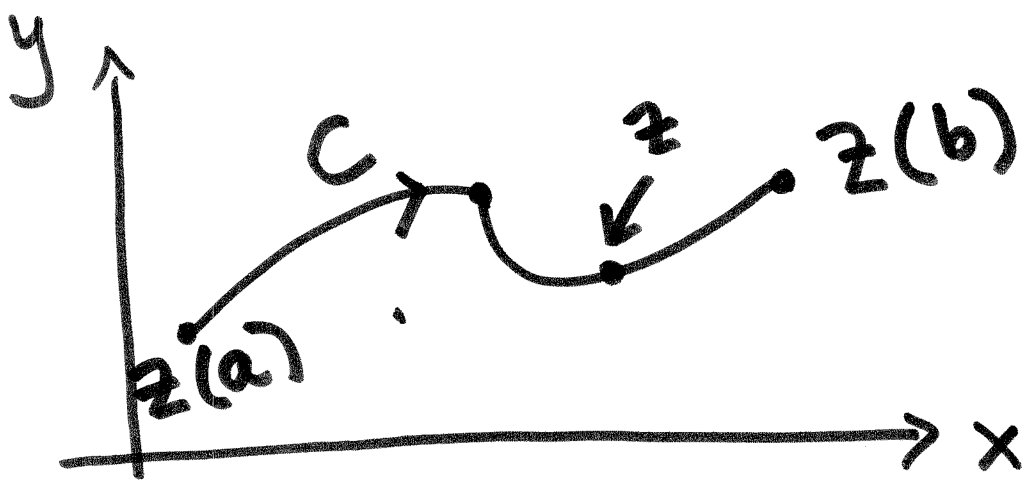
Contour/Line Integrals

Goal: Evaluate $\int_C f(z)\,dz$

$C: \; z(t) = x(t) + i\,y(t)$

$a \le t \le b$



z-plane

Contour/Line Integrals

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Contour : A piecewise smooth curve $z(t)$; $a \le t \le b$

$$z(t) = x(t) + i\,y(t)$$

or, a number of smooth curves joined back to back.

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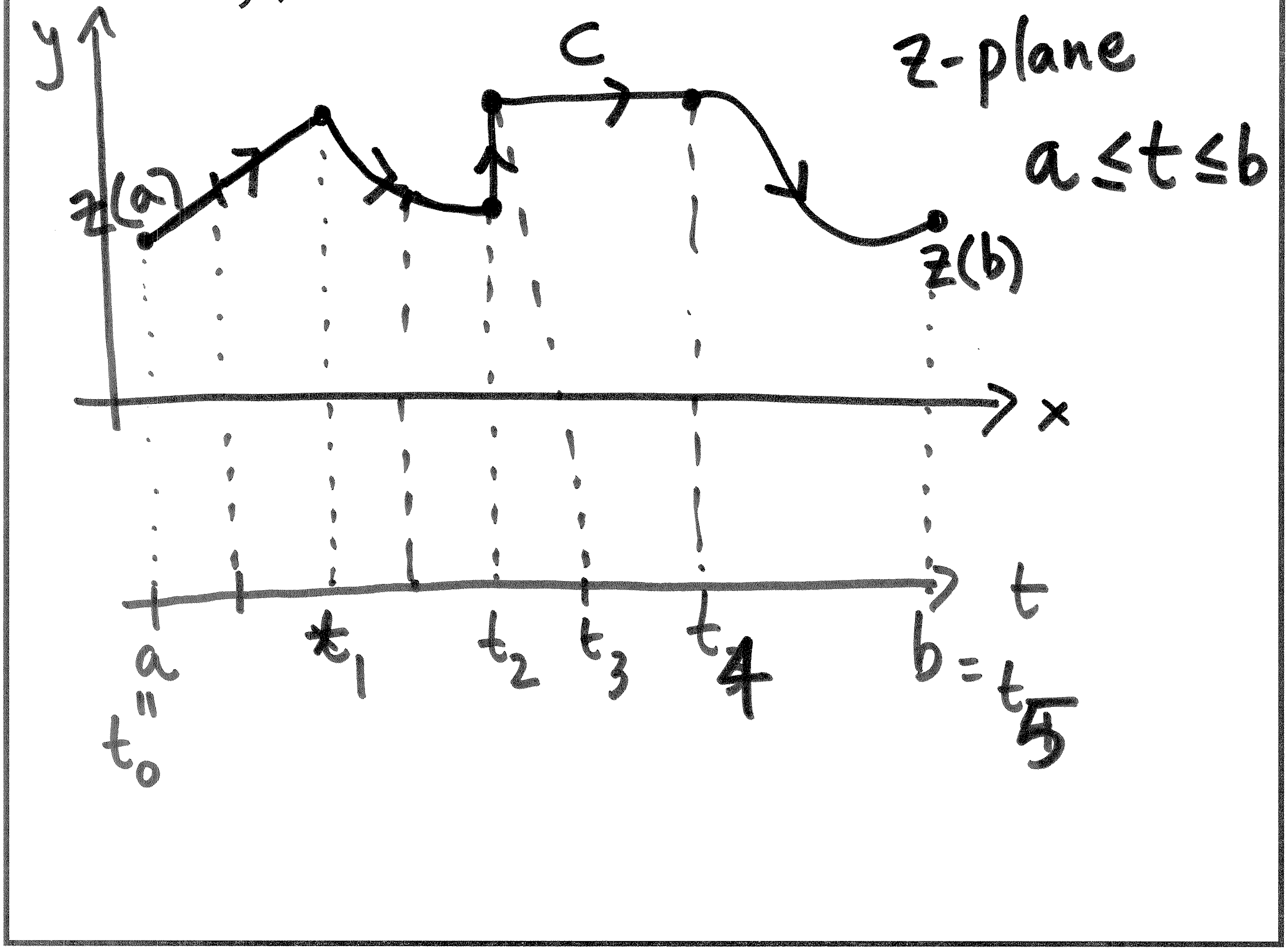

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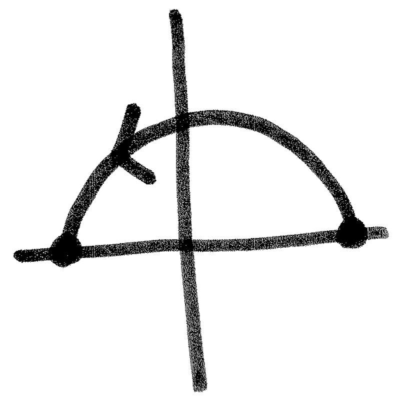
<u>Example</u>

$$\int_C \overbrace{z^2|z|}^{f}\, dz$$

a) C : upper half of the circle $|z|=2$; counterclockwise

$C:\quad z = 2e^{it}$, $0 \le t \le \pi$



$$dz = 2ie^{it}\, dt$$

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$$\int_C z^2 |z|\, dz = \int_0^{\pi} 4e^{i2t} \cdot 2 \cdot 2ie^{it}\, dt$$

$$= 16i \int_0^{\pi} e^{i3t}\, dt$$

$$= \frac{16i}{3i} e^{i3t} \Big|_0^{\pi}$$

$$= \frac{16}{3} \left(e^{i3\pi} - 1\right) = -\frac{32}{3}$$

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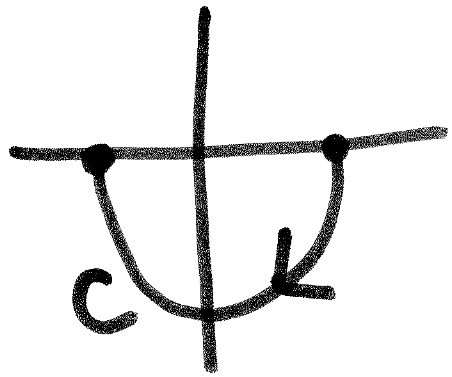
$$\int_C z^2|z|\,dz$$

b) C: the lower half of $|z|=2$; clockwise.

$C: \; z = 2e^{it}$



$2\pi \le t \le \pi$

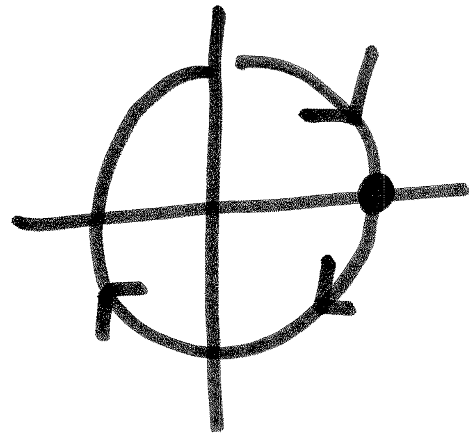
$$\int_C z^2|z|\,dz = \int_{2\pi}^{\pi} 4e^{i2t}\, 2 \cdot 2ie^{it}\,dt \; \cdots = -\frac{32}{3}$$

6 University of Idaho $\int_C z^2 |z| \, dz$

c) C: the circle $|z| = 2$ clockwise

$z = 2e^{it}$

$dz = 2ie^{it} dt$



$$\int_{2\pi}^{0} 4e^{i2t} \cdot 2 \cdot 2ie^{it} \, dt$$

$= 0$

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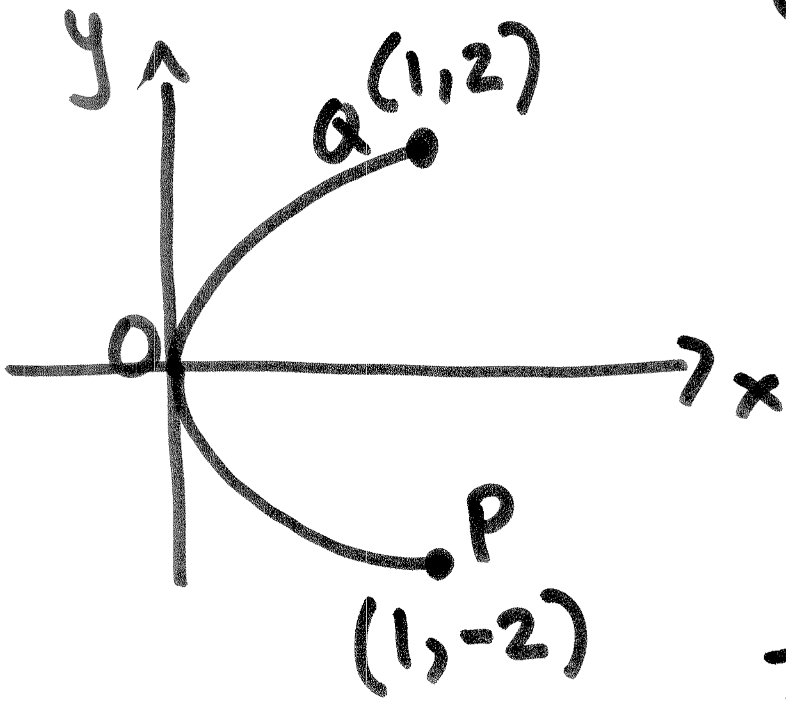
Example (HW) $\int_C (z-2)\,\bar{z}\,dz$



C: the arc POQ of the parabola

$$y^2 = 4x$$

$$z = t^2 + i\,\underbrace{2t}_{y}$$

$$-1 \le t \le 1$$

Properties of contour integrals

a) $\int_C f(z)\,dz$

$-C$: has the same points as C but moving in the opposite direction

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$$

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$$C = C_1 + C_2 + C_3$$

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b)

OA : C_1

AB : C_2

BD : C_3

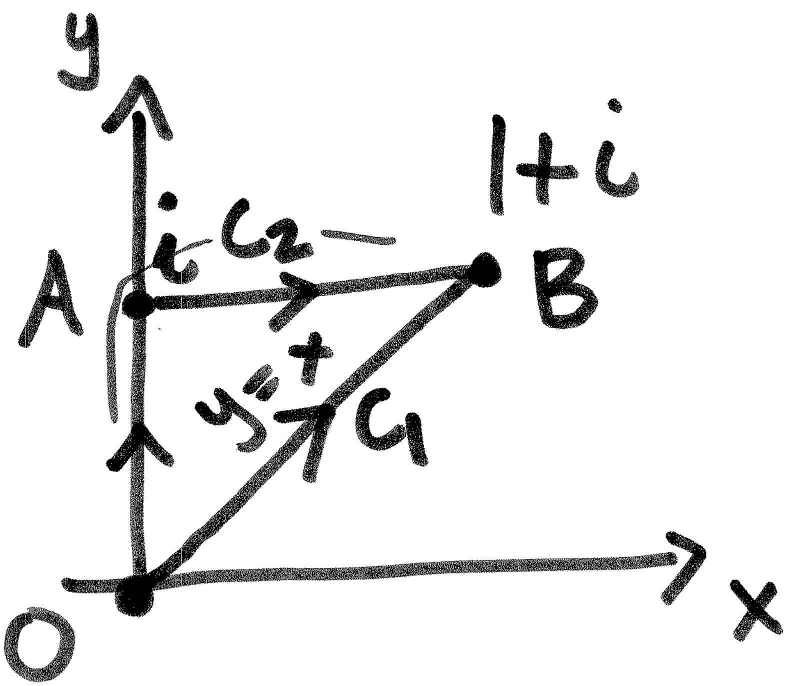
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz$$

c)

$$\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$$

Integral depends on the path

Ex



$f(z) = y - x - i3x^2$

$C_1 = OB$:

$z = t + it$

$0 \le t \le 1$

$dz = (1+i)dt$

$$I_1 = \int_{C_1} f(z)\,dz = \int_0^1 (\cancel{t} - \cancel{t} - i3t^2)(1+i)dt$$

$$= 1 - i$$

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$$C_2 = OA + AB$$

$OA: \ z = it, \ 0 \le t \le 1$

$dz = i\,dt$

$AB: \ z = t + i, \ 0 \le t \le 1$

$dz = dt$

$$\int_{C_2} f(z)\,dz$$

$$= \int_{OA} + \int_{AB}$$

$$= \int_0^1 t\,i\,dt + \int_0^1 (1 - t - i3t^2)\,dt$$

$\vdots$

$$= \frac{1-i}{2}$$