MATH 420

COMPLEX VARIABLES

SESSION no. 16

Topics for today

- Upper bound for the absolute value of a complex line integral

- Cauchy's Integral Theorem

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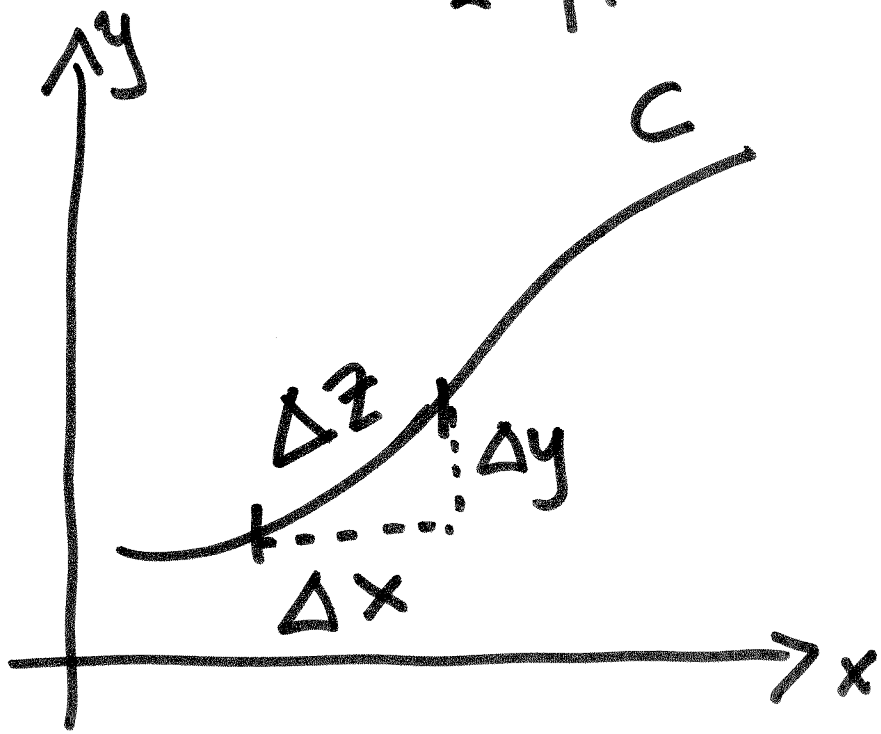
Triangle Inequality:

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots |z_n|$$

$$\left| \int_C f(z)\, dz \right| \leq \int_C |f(z)|\, dz$$

Arc - length

z- plane



$$|\Delta z| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$z = z(t)$$

$$a \le t \le b$$

$$|z'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$$

$$\text{Arc Length} = \int_a^b |z'(t)|\,dt$$

Theorem: C: a contour with length L

Let $f(z)$: continuous on C

$|f(z)| \leq M$ for $z \in C$

Then

$$\left| \int_C f(z)\,dz \right| \leq ML$$

Proof: $z = z(t)$

$$\left| \int_C f(z)\,dz \right| \leq \int_C |f(z)|\,dz$$

$$= \int_C |f(z)|\,|z'(t)|\,dt$$

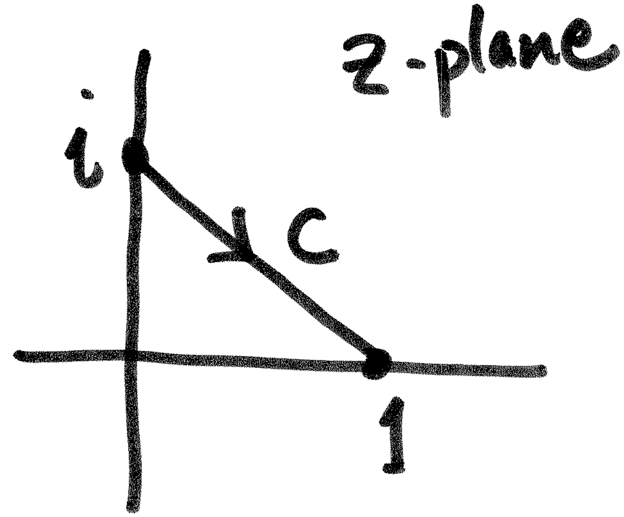
$$\leq M \int_C |z'(t)|\,dt \rightarrow \text{arc length}$$

$$= M L$$

Example: C : the line from $z = i$ to $z = 1$

Show that

$$\left| \int_C \frac{dz}{z^4} \right| \le 4\sqrt{2}$$



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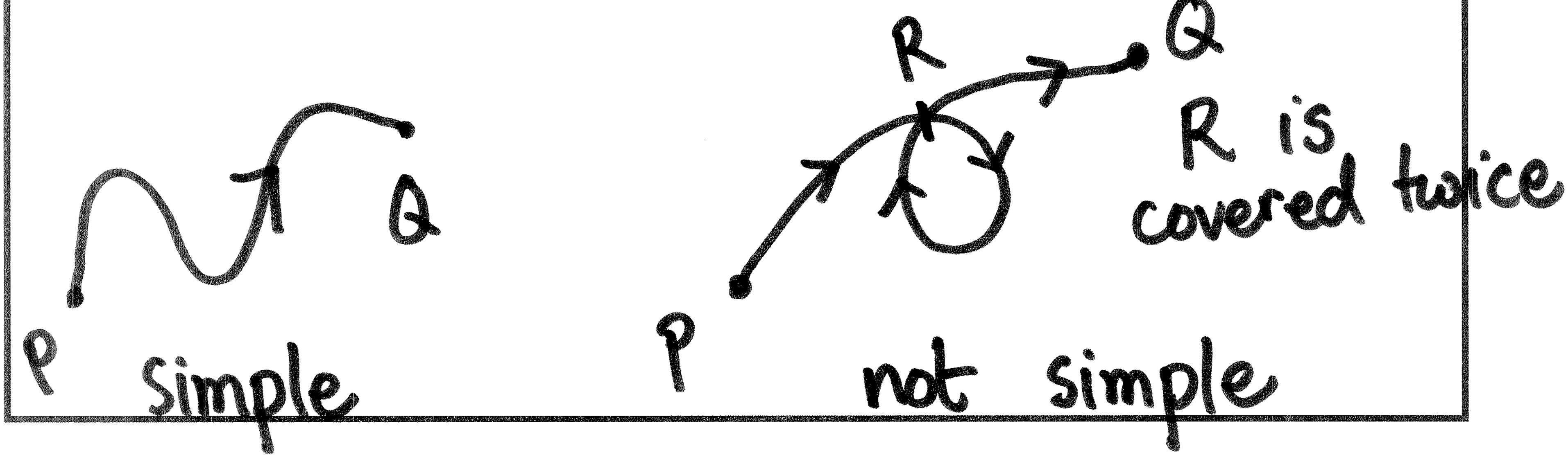
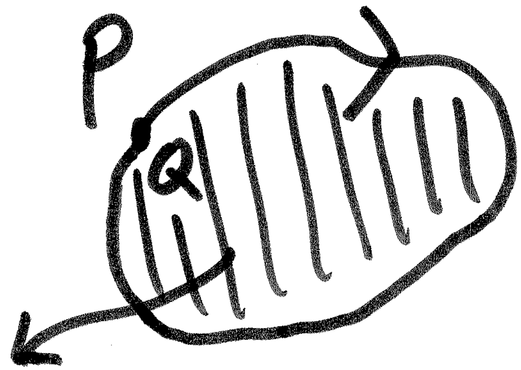
Cauchy's Integral Thm.

1) Simple curve: a path between P and Q is a simple curve if each point on the path is traversed only once.



Simple closed curve: A simple curve whose end points P & Q coincide.



Simply connected domain: The interior of a simple closed curve.

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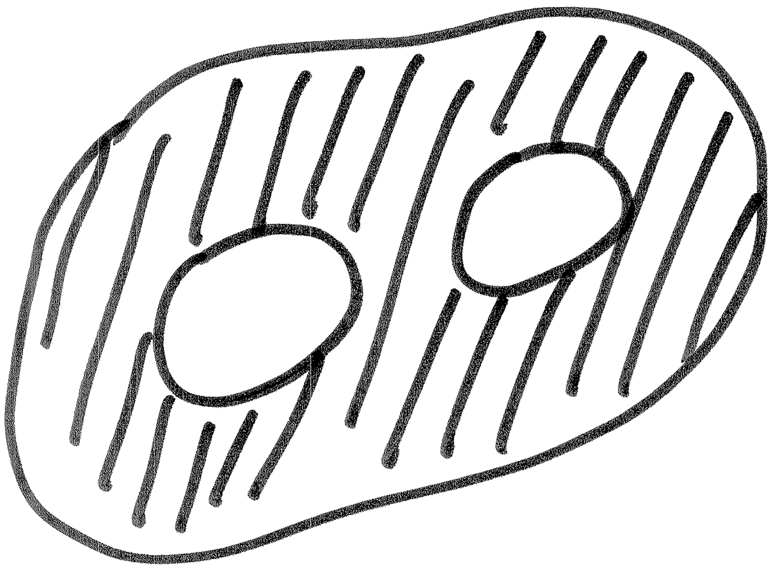


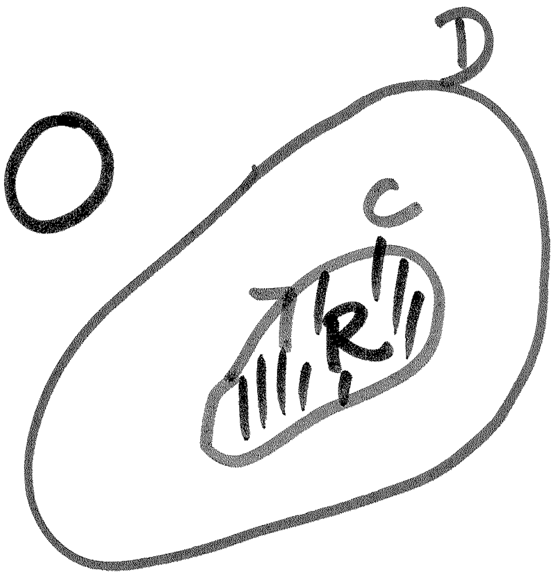
3-fold connected

<u>If</u> the boundary has p distinct simple closed curves then the domain is p-fold connected.

Cauchy's Integral Theorem:

Let $f(z)$ be an analytic in a simply connected domain D. Then for any closed path C in D

$$\int_C f(z)\,dz = 0$$



Proof:

Recall: Green's Theorem (from Calc III)

$$\int_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$$\int_C u\,dx - v\,dy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$f = u + iv$ analytic $\Longrightarrow \begin{cases} \partial u/\partial x = \partial v/\partial y, \quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \\ \text{cont. partial derivatives} \end{cases}$

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$Z = x + iy$

$\cancel{\oint_C f(z)\,dz}$ $f(z)\,dz$

$= (u+iv)(dx+idy)$

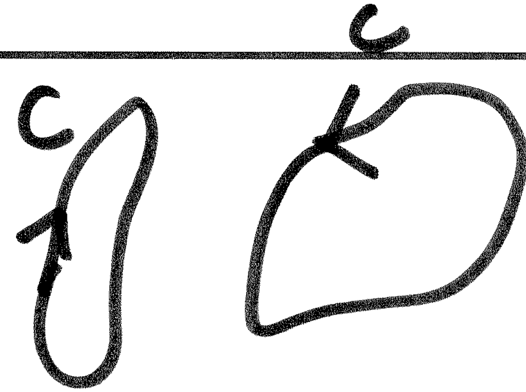
$= (u\,dx - v\,dy) + i(v\,dx + u\,dy)$

$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$= 0 + 0 = 0$

Ex: $\displaystyle\int_C e^{z^3}\,dz$



C: is any simple closed path

e^{z^3}: is analytic <u>everywhere</u>

By Cauchy's Thm $\displaystyle\int_C e^{z^3}\,dz = 0$