MATH 420

COMPLEX VARIABLES

SESSION no. 17

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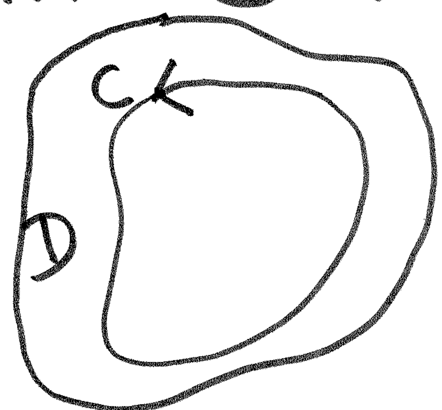
Last lecture:

Cauchy's Integral Theorem -

If $f(z)$ is analytic in a simply connected domain D then

$$\int_C f(z)\,dz = 0$$



for any closed path C in D.

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Today :

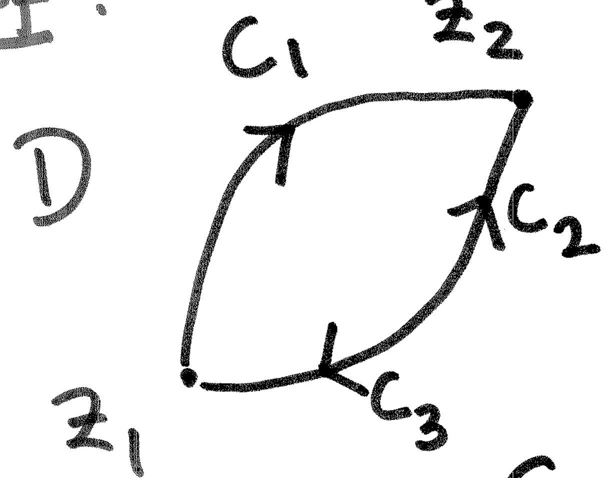
- Several consequences of Cauchy's Integral Theorem
- Cauchy's Integral Formula

Consequences of Cauchy's Int. Thm.

a) Let f be analytic in a simply connected domain D. If $z_1, z_2 \in D$

$$\int_{z_1}^{z_2} f(z)\,dz \quad \text{is independent of the path.}$$

<u>Proof</u>:



$$C = C_1 + C_3$$

$$0 = \int_C f = \int_{C_1} + \int_{C_3}$$

Cauchy's Int. Thm

4

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$$0 = \int_C = \int_{C_1} + \int_{C_3}$$

$$= \int_{C_1} - \int_{C_2}$$

$$\Longrightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

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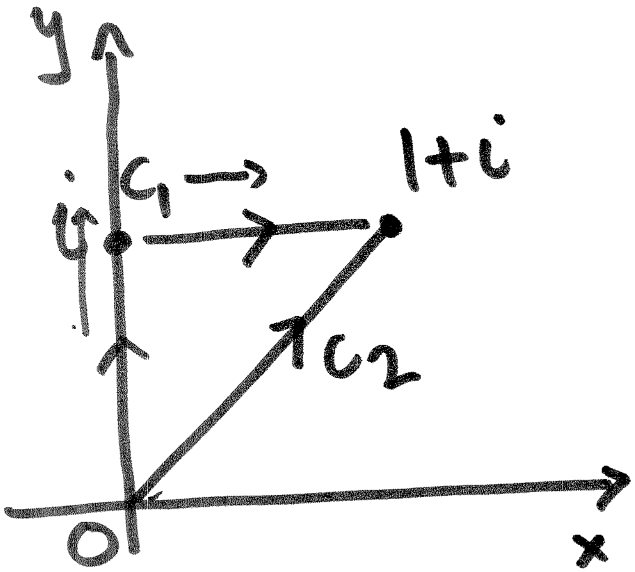
$z = x + iy$

Example : $f(z) = y - x - i\,3x^2$



$$\int_{C_1} f(z)\,dz = \frac{1-i}{2}$$

$$\neq$$

$$\int_{C_2} f(z)\,dz = 1-i$$

$f = u + iv \Rightarrow u = y - x \,,\; v = -3x^2$

$$\frac{\partial u}{\partial x} = -1 \neq \frac{\partial v}{\partial y} = 0 \,,\quad \frac{\partial u}{\partial y} = 1 \quad \frac{\partial v}{\partial x} = -6x$$

$$\partial u/\partial y = -\frac{\partial v}{\partial x} \Rightarrow x = 1/6$$

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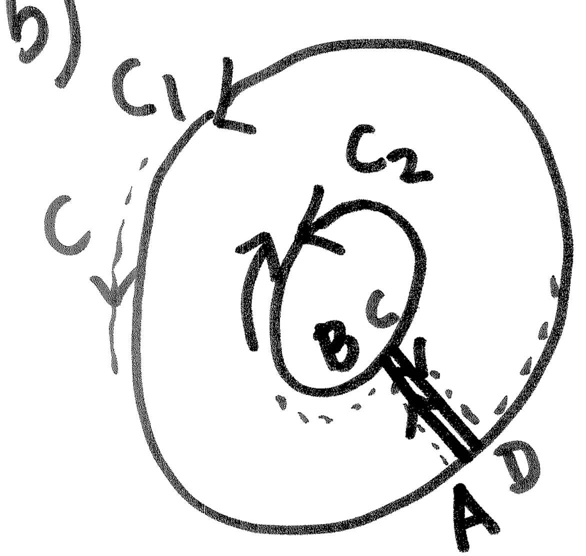
b)



If f is analytic in the annular region between C_1 & C_2 then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Proof: $C = C_1 + AB - C_2 + CD$

Then C is a simple closed path, f is analytic in the region bounded by C.

7

Cauchy's Int. Thm

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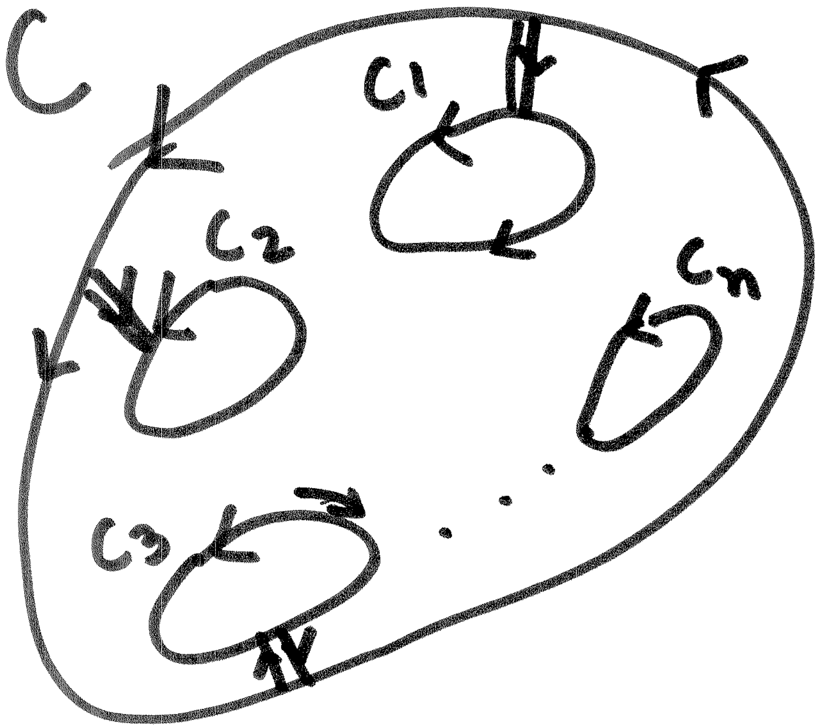
equal & opposite

$$0 = \int_C f(z)\,dz = \int_{C_1} + \int_{AB} - \int_{C_2} + \int_{CD}$$

$$= \int_{C_1} - \int_{C_2}$$

$$\Longrightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Generalization:



$$\int_C f(z)\,dz = \int_{C_1} + \int_{C_2} + \cdots + \int_{C_n}$$

Example: Evaluate $\int_C \frac{dz}{z-a}$

C is any simple closed curve

a) $z = a$ is outside C

b) $z = a$ is inside C.

$f(z) = \frac{1}{z-a}$ not analytic at a.

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a)

$$\int_C \frac{dz}{z-a} = 0$$

by Cauchy's Integral Thm

since f is analytic everywhere except $z = a$

11

$f(z) = \int_C dz/z-a$

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b)

γ : circle centered at a, radius ε

$$\int_C \frac{dz}{z-a} = \int_\gamma \frac{dz}{z-a}$$

$$\gamma : z(t) = a + \varepsilon e^{it} \quad 0 \le t < 2\pi$$

$$dz = \varepsilon i e^{it} dt$$

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$$\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{\cancel{\varepsilon} i \cancel{e^{it}}}{\cancel{\varepsilon} \cancel{e^{it}}}\, dt$$

$$= i \int_0^{2\pi} dt = i\,2\pi$$

$$\Rightarrow \int_C \frac{dz}{z-a} = 2\pi i$$

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Show that

$$\int_C \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i & n=1 \\ 0 & n\neq 1 \end{cases}$$

C : simple closed path around the point a.