MATH 420

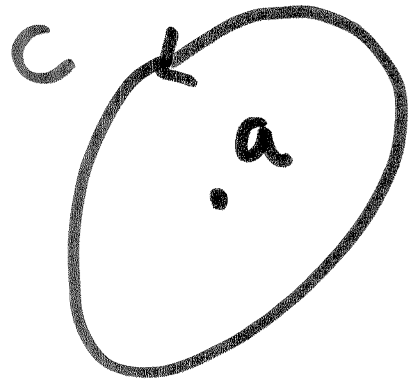
COMPLEX VARIABLES

SESSION no. 18

Recall from last lecture:

f is analytic

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz$$



$$\int_C \frac{dz}{z-a} = 2\pi i$$

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Cauchy's Integral Formula

f : analytic on and inside a simple closed curve C containing a, then

$$2\pi i\, f(a) = \int_C \frac{f(z)}{z-a}\,dz$$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

$\frac{f(z)}{z-a}$ is not analytic at $z=a$

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C_1 : circle with center a ; radius (small) r

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz$$

$$= \int_{C_1} \frac{f(a) + f(z) - f(a)}{z-a}\,dz$$

$$= f(a)\int_{C_1} \frac{dz}{z-a} + \int_{C_1} \frac{f(z)-f(a)}{z-a}\,dz$$

$$= f(a)\,2\pi i + 0$$

due to continuity of f at a

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f is cont. at $z=a \Rightarrow$ given ε, $\exists r$

s.t. $|f(z)-f(a)|<\varepsilon$, $|z-a|\leq r$

$$\left|\frac{f(z)-f(a)}{z-a}\right| = \frac{|f(z)-f(a)|}{r} < \frac{\varepsilon}{r}$$

$$\left|\int_{C_1} \frac{f(z)-f(a)}{z-a}\,dz\right| \leq \frac{\varepsilon}{r}\,\text{arclength}(C_1)$$

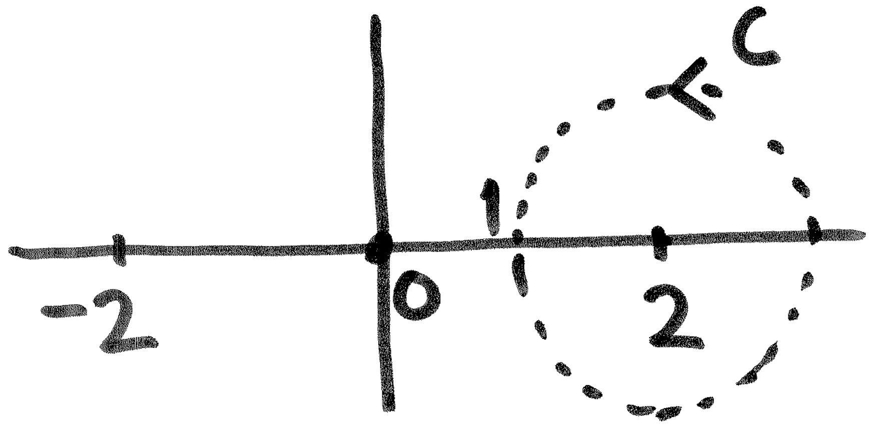
$$= \frac{\varepsilon}{r}\,2\pi r = 2\pi\varepsilon$$

can be made arbitrarily small with ε

Integral Formula

$$\int_C \frac{f(z)}{z-a}\,dz = f(a)\,2\pi i$$

Ex $\displaystyle\int_C \frac{2z}{z^2-4}\,dz$ C : is the unit circle, center $z=2$



$$\frac{2z}{z^2-4} = \frac{2z}{(z-2)(z+2)}$$

$f(z)$

not analytic at $z=2$

6

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By Cauchy's Formula: $f(z) = \frac{2z}{z+2}$

$$2\pi i\, f(2) = \int_C \frac{f(z)}{z-2}\,dz = \int_C \frac{2z}{z^2-4}$$

$$f(2) = \frac{4}{4} = 1$$

$$\Rightarrow \int_C \frac{2z}{z^2-4} = 2\pi i$$

C: circle centered at $z = 2$

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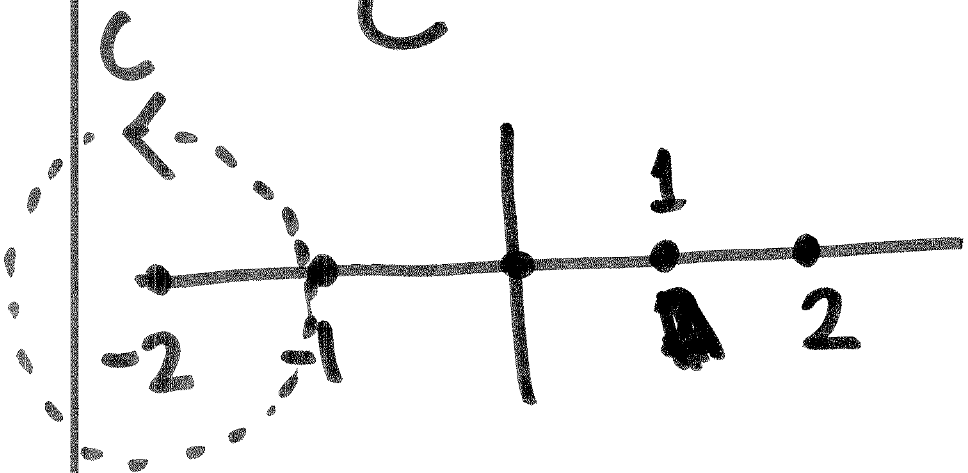
Ex $\int_C \frac{2z}{z^2-4}\,dz$

C: unit circle centered at $z = -2$



$$\frac{2z}{z^2-4} = \frac{2z}{(z-2)(z+2)}$$

f

not analytic at $z = -2$

$$\int_C \frac{2z}{z^2-4}\,dz = 2\pi i\, f(-2) = 2\pi i$$

$f(-2) = 1$

8

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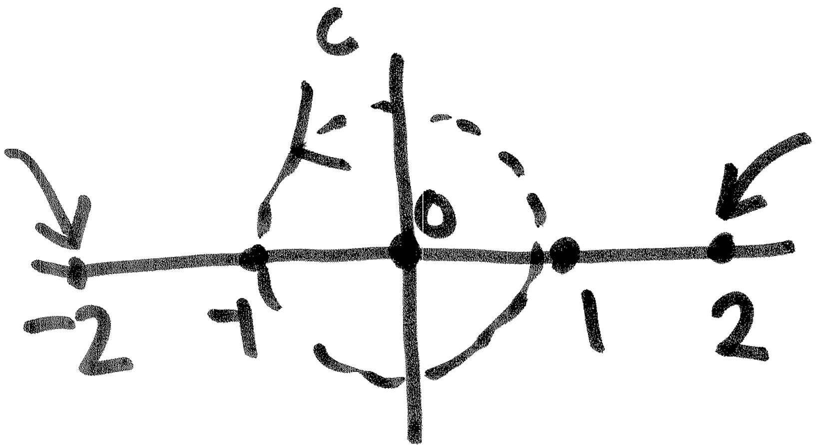
$$\int_C \frac{2z}{z^2-4}\,dz$$

C: a unit circle centered at $z=0$



$$\int_C \frac{2z}{z^2-4}\,dz = 0$$

Cauchy's Thm.

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Consequence of the Integral Formula

f: analytic on and inside C

C contains a point a

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$

$$\frac{d}{da} f(a) = \frac{1}{2\pi i} \int_C \frac{\cancel{(-1)^2} f(z)}{(z-a)^2}\, dz$$

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$$\frac{d^2}{da^2} f(a) = f''(a) = \frac{1}{2\pi i} \int_C \frac{f(z)(-2)(-1)}{(z-a)^3}$$

$$= \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3} dz$$

$$\vdots$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

Derivatives of all orders exist

11

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In the real case

$$f(x) = (x-1)^{3/2}$$

$$f'(x) = \frac{3}{2}(x-1)^{1/2}$$ exist at $x=1$

$$f''(x) = \frac{3}{4}\frac{1}{(x-1)^{1/2}}$$ does not exist at $x=1$

Ex $\displaystyle\int_C \frac{e^{2z}}{z^4}\,dz$

C : unit circle ; center at $z=0$

$\displaystyle\int_C \frac{e^{2z}}{z^{3+1}}$ set $f(z)=e^{2z}$, $a=0$, $n=3$

$f'(z) = 2e^{2z}$

$= \dfrac{2\pi i}{3!} f^{(3)}(0)$

$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

$= \dfrac{2\pi i}{6}\, 8 = \dfrac{8\pi i}{3}$

$f'''(0) = 8$