MATH 420

COMPLEX VARIABLES

SESSION no. 19

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Previously: Cauchy's Int. For.

curve

C: simple closed path about a

$f(z)$: analytic on and inside C

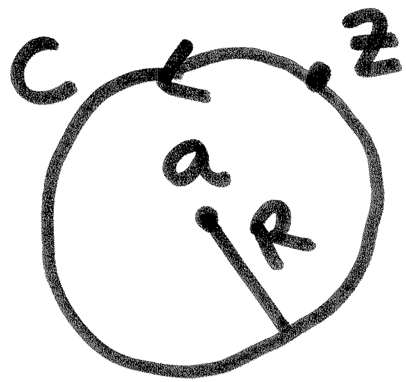
$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Suppose:

C: circle of radius R, center a

$|f(z)| \le M$ for $z \in C$



$$|f^{(n)}(a)| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz\right|$$

$$\le \frac{n!}{2\pi}\,\frac{M}{R^{n+1}}\,2\pi R$$

$$= n!\,M/R^n$$

$|z-a| = R$

$$\left|\frac{f(z)}{(z-a)^{n+1}}\right| \le \frac{M}{R^{n+1}}$$

arc length C = $2\pi R$

$$\left|f^{(n)}(a)\right| \le \frac{n!\,M}{R^n}$$

Cauchy's estimate

4

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Liouville's Theorem: Let $f(z)$ ~~is~~ be analytic for all z (f is entire) and $|f(z)| \leq M$ (f is bounded) then $f(z)$ is a constant.

Proof: Let C: circle of radius R

$0 \leq |f'(z)| \leq \frac{M}{R}$ Let $R \to \infty$

$\Rightarrow |f'(z)| = 0 \Rightarrow f'(z) = 0 \Rightarrow f$ is a constant

<u>Example</u> : $f(x) = \sin(x)$, x is real.

1) $\sin x$ is analytic everywhere
2) $|\sin x| \leq 1$, bounded

but $\sin x$ is not constant.

Does not contradict Liouville's theorem because $\sin z$ is not bounded for z complex

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

$$a_n \neq 0$$

$P(z)$ is a polynomial of degree n

Any solution of

$$P(z) = 0$$

is a root of $P(z)$.

7

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$$P(z) = z^2 + 1 = 0$$

$$\Rightarrow z^2 = -1$$

$$\Rightarrow z = \pm\sqrt{-1}$$

$$= \pm i$$

$P(z)$ has not roots in $\mathbb{R}$ but has roots in the complex no. system.

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Fundamental Thm. of Algebra

Every polynomial equation

$P(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$, $n \geq 1$

has at least one root.

<u>Proof</u>: Assume that $P(z) \neq 0$ for any z. $f(z) = \dfrac{1}{P(z)}$ is analytic everywhere. $f(z)$ is bounded; $|f(z)| \to 0$ as $|z| \to \infty$

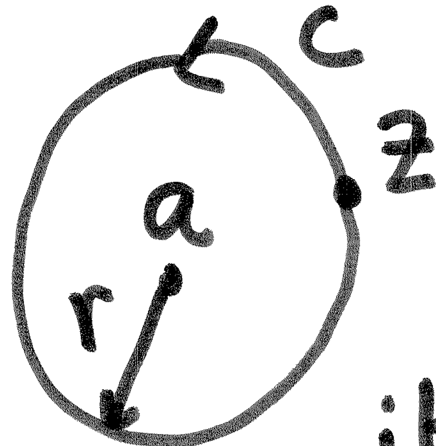
By Liouville's theorem $f(z)$ must be a constant — a contradiction $\Rightarrow$ $P(z)$ must have at least one root.

Gauss Mean Value Theorem:

C: center at a, radius r

f(z) is analytic on and inside C.

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$



$$z(t) = a + re^{it}$$

$$0 \le t < 2\pi$$

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$$dz = r i e^{it} dt$$

$$f(a) = \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(a+re^{it})}{\cancel{r} \cancel{e^{it}}} \cancel{r} \cancel{i} \cancel{e^{it}} dt$$

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{it}) dt$$

Gauss Mean Value Thm