MATH 420

COMPLEX VARIABLES

SESSION no. 20

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Last Lecture

Important theorems

- Liouville's Theorem
- Fundamental Theorem of Algebra
- Gauss Mean Value Theorem

Today

- Gauss Mean Value Theorem
- Maximum Modulus Theorem

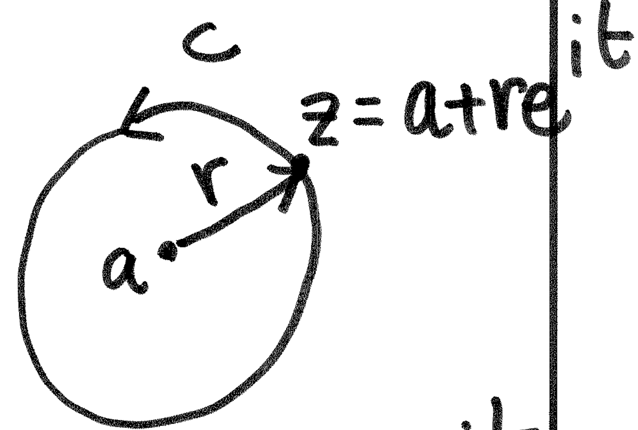
* Infinite Series representation of analytic functions

Gauss Mean Value Thm.

C: center at $z = a$

f is analytic on and inside C

$$\boxed{f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(\overbrace{a+re^{it}}^{z})\,dt}$$



value of f at the center = mean of f on C

$z(t) = a + re^{it}$

$0 \le t < 2\pi$

Ex: Find the mean value of $x^2 - y^2 + 2y$ over the circle

f

$$|z - 5 + 2i| = 3.$$

$$\downarrow$$

$$|z - (5 - 2i)| = 3$$

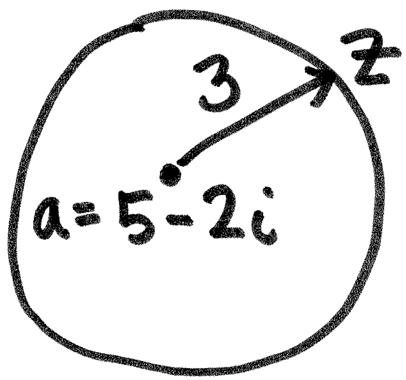
Mean value

$= f(5 - 2i)$

$= 5^2 - (-2)^2 + 2(-2)$

$= 25 - 4 - 4$

$= 17$



$x = 5$

$y = -2$

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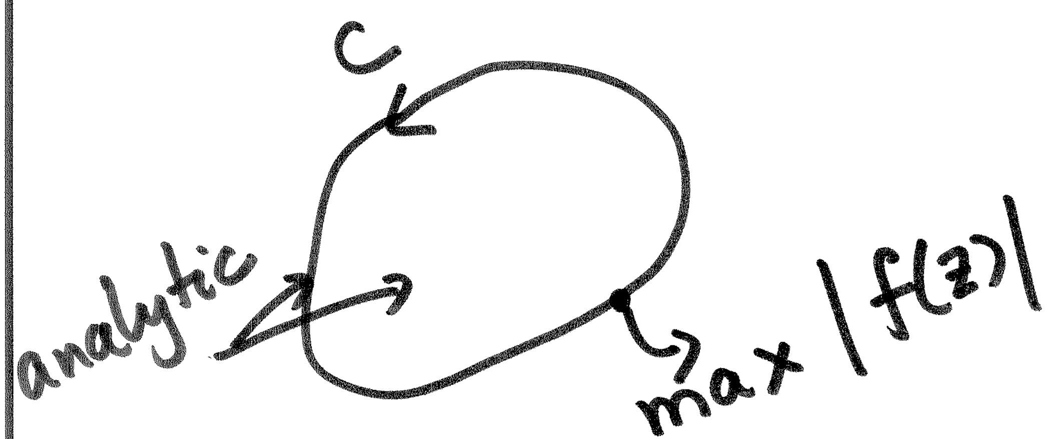
Maximum Modulus Theorem

Suppose $f(z)$ is analytic on and inside a simple closed curve C.

Then $|f(z)|$ is attained on C, unless f is a constant



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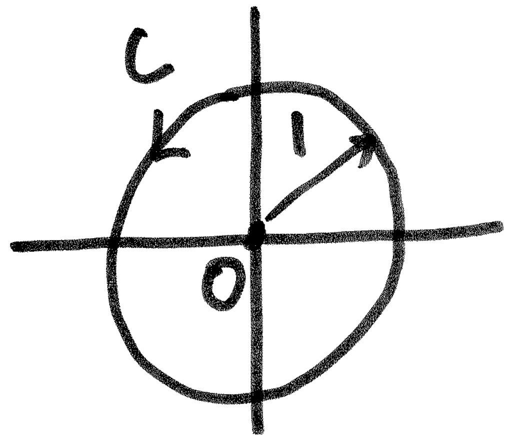
not analytic

Example $\quad f(z) = 2 - |z|^2$

C: boundary of the unit circle
$|z| = 1$.

On C: $|z| = 1$

$\Rightarrow \quad |f(z)| = 1$



At $z = 0$, $|f(z)| = 2$

$$1 \leq |f(z)| \leq 2$$

max$|f(z)|$ is NOT attained on C

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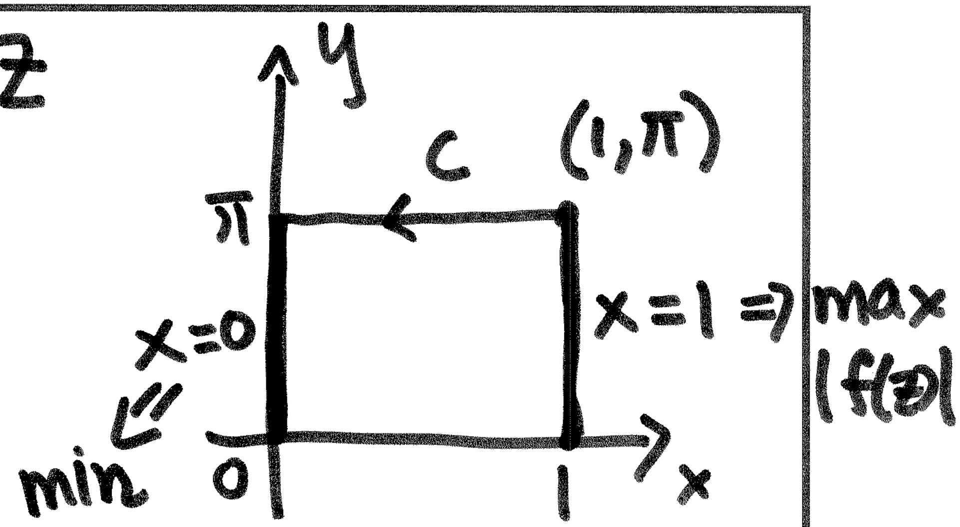
Ex : $f(z) = e^{z}$



$$f(z) = e^{x+iy}$$

$$= e^{x} e^{iy}$$

$$\Rightarrow |f(z)| = e^{x}, \quad 0 \le x \le 1$$

max $|f(z)|$ at $x = 1$.

min $|f(z)|$ at $x = 0$.

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Minimum modulus : f is analytic on and inside a closed curve C then $\min |f(z)|$ is attained on C, unless $f(z) = 0$ inside C.

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Infinite Series

The sum $\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + \cdots$

is called an infinite series.

Partial sums:

$$s_n = z_1 + z_2 + \cdots + z_n$$

is the n-th partial sum.

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Convergence of series

The series $\sum_{k=1}^{\infty} z_k$ converges if and only if $\{s_n\}_{n=1}^{\infty}$ also converges.

<u>Ex</u>: $\overset{z_1}{1} + \overset{z_2}{(1+i)} - \overset{z_3}{(1+i)} + \overset{z_4}{(1+i)} + \cdots$

$s_1 = 1$, $s_2 = 1 + (1+i) = 2+i$

$s_3 = 1 + (1+i) - (1+i) = 1$, $s_4 = 1 + (1+i) - (1+i) + (1+i) = 2+i$

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$$\{s_n\}_{n=1}^{\infty} = \{1,\ 2+i,\ 1,\ 2+i,\ \dots\}$$

does not converge (oscillating)

$\Rightarrow\ 1+(1+i)-(1+i)+\cdots$ does not converge.

Geometric series: $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$

converges if and only if $|z| < 1$

and then $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

Ex: $1 + \frac{i}{2+i} - \frac{1}{(2+i)^2} + \cdots + \frac{i^n}{(2+i)^n} + \cdots$

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$$z = \frac{i}{2+i}, \quad |z| = \frac{1}{|2+i|} = \frac{1}{\sqrt{5}} < 1$$

$$\Rightarrow 1 + \frac{i}{2+i} + \cdots + \frac{i^n}{(2+i)^n} \text{ converges}$$

Absolute convergence : For $\sum_{n=1}^{\infty} z_n$

if $\sum_{n=1}^{\infty} |z_n|$ converges then $\sum z_n$

is said to converge absolutely.

Ex: $1 + \frac{i}{2+i} - \frac{1}{(2+i)^2} + \cdots + \frac{i^n}{(2+i)^n} + \cdots$

$\sum |z_n| = 1 + \frac{1}{\sqrt{5}} + \left(\frac{1}{\sqrt{5}}\right)^2 + \cdots$

converges because $1/\sqrt{5} < 1$

$\Rightarrow$ the given series is absolutely convergent.

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Ex: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

converges but the series

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ (harmonic series)

does not converge. The given series converges but is not absolutely convergent.