# MATH 420

# COMPLEX VARIABLES

## SESSION no. 21

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# Infinite Series

Convergence: $$\sum_{n=1}^{\infty} z_n$$

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| < 1 \Rightarrow \text{absolute conv.}$$

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| > 1 \Rightarrow \text{divergence}$$

$$\text{"} \quad = 1 \Rightarrow \text{inconclusive}$$

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$$\sum_{n=0}^{\infty} z_n$$

Root test:

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} < 1 \Rightarrow \text{absolute conv.}$$

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} > 1 \Rightarrow \text{divergence}$$

$$= 1 \Rightarrow \text{inconclusive}$$

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Ex. $$\sum_{n=0}^{\infty} \frac{(3+i4)^n}{n!}$$

Ratio test:

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = \lim_{n\to\infty} \left| \frac{(3+i4)^{n+1}}{(n+1)!} \frac{n!}{(3+i4)^n} \right|$$

$$= \lim_{n\to\infty} \frac{|3+i4|}{n+1} = \lim_{n\to\infty} \frac{5}{n+1} = 0 < 1$$

$\Rightarrow$ series converges absolutely

# Power Series

A series of the form

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

is called a power series about $z_0$.

$z_0$ : a fixed no. , $z$ - variable

The series may converge for certain values of $z$ and diverge for others.

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$(z-i)^n$

**Ex:** $\displaystyle\sum_{n=0}^{\infty} \frac{n!\, z^n}{2^n}$ Find the set for which this converges.

$(z-i)^{n+1}$

Ratio Test: $\displaystyle\lim_{n\to\infty}\left|\frac{(n+1)!\, z^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n!\, z^n}\right|$

$(z-i)^n$

$|z-i|$

$$= \lim_{n\to\infty} \frac{(n+1)}{2}|z| = \begin{cases} 0 & \text{if } z=0 \\ \infty & \text{if } z\neq 0 \end{cases}$$

$z=i$

$\neq i$

The series converges at $z = 0$ only

We say, the radius of convergence $= 0$

Ex: $$\sum_{n=0}^{\infty} \frac{(z-2i)^n}{(n+1)(n+2)}$$

Ratio test: $$\lim_{n\to\infty} \left| \frac{(z-2i)^{n+1}}{\cancel{(n+2)}(n+3)} \frac{(n+1)\cancel{(n+2)}}{(z-2i)^n} \right|$$

$$= \lim_{n\to\infty} \overbrace{\frac{n+1}{n+3}}^{\to 1} |z-2i| = |z-2i|$$

Convergence for $|z-2i| < 1$

$\Rightarrow$ Radius of convergence $= 1$

$\Rightarrow$ the set

$|z-2i| < 1$ is

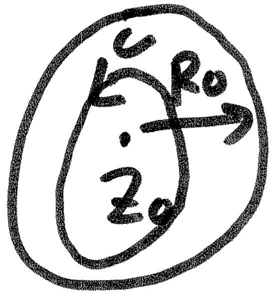
called the disk of convergence.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

For such functions (as $e^z$) that converge everywhere in the z-plane, the radius of convergence $= \infty$.

# Taylor Series

**Theorem**: Suppose that $f$ is an analytic function over the disk $|z - z_0| < R_0$, then $f$ has the power series rep.




$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{\cancel{n!}} \frac{\cancel{n!}}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

$$= \frac{1}{2\pi i} \int_C f/(z - z_0)^{n+1}$$

Every power series

$$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

is analytic.

If $z_0 = 0$, the series is called the Maclaurin series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

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University of Idaho ~~Maclaurin~~

Find the ~~Taylor~~ series of $z^2 e^{3z}$.

Know that $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

1) Replace $z$ by $3z$: $1 + 3z + \frac{(3z)^2}{2!} + \cdots$

$$e^{3z} = \sum_{n=0}^{\infty} (3z)^n / n!$$

$f = z^2 e^{3z}$



2) Multiply by $z^2$:

$$z^2 e^{3z} = \sum_{n=0}^{\infty} \frac{(3z)^n z^2}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{3^n z^{n+2}}{n!}$$

same

3) Renumber: $$\sum_{n=2}^{\infty} \frac{3^{n-2} z^n}{(n-2)!}$$

Next: Series exp. for non-analytic fns.