MATH 420

COMPLEX VARIABLES

SESSION no. 22

Last class:

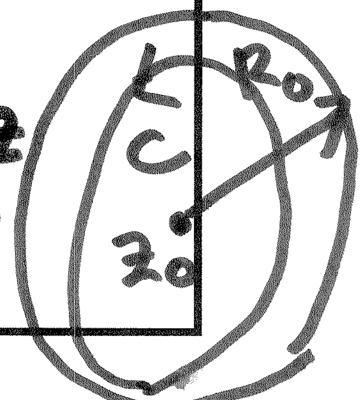
University of Idaho

Taylor's Theorem: If f is analytic throughout $|z - z_0| < R_0$ then

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \longleftarrow \text{Taylor series of } f \text{ about } z_0$$

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{n!}\,\frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz$$



$$a_n = \frac{1}{2\pi i} \int_C f/(z - z_0)^{n+1}$$

University of Idaho

Maclaurin series if $z_0 = 0$

1. $$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

2. $$\sin z = z - z^3/3! + z^5/5! - z^7/7! + \cdots$$

3. $$\cos z = 1 - z^2/2! + z^4/4! - z^6/6! + \cdots$$

4. $$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \; ; \quad |z| < 1$$

5. Replace z by $-z$ in 4.

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$$

6. Replace z by z^2 in 5.

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$$

Ex. Find the Taylor series of e^z about $z=1$, i.e., write e^z as $a_0 + a_1(z-1) + a_2(z-1)^2 + \cdots$

find

$$e^z = e\,e^{z-1} = e\left(1 + (z-1) + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} \cdots\right)$$

$$= \underbrace{e}_{a_0} + \underbrace{e}_{a_1}(z-1) + \underbrace{\frac{e}{2!}}_{a_2}(z-1)^2 + \frac{e}{3!}(z-1)^3 + \cdots$$

Ex: Find the Taylor series of $f(z) = \sin z$ about $z = \pi/4$

$$\sin z = a_0 + a_1\left(z - \frac{\pi}{4}\right) + a_2\left(z - \pi/4\right)^2 + \cdots$$

$$a_n = \frac{f^{n}(\pi/4)}{n!}$$

$f(z) = \sin z \; ; \; f(\pi/4) = 1/\sqrt{2}$

$f'(z) = \cos z \; ; \; f'(\pi/4) = 1/\sqrt{2}$

$f''(z) = -\sin z \; ; \; f''(\pi/4) = -1/\sqrt{2}$

$f'''(z) = -\cos z \; ; \; f'''(\pi/4) = -1/\sqrt{2}$

$\vdots$

University of Idaho

$$\sin z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right) - \frac{1}{2!\sqrt{2}}\left(z-\frac{\pi}{4}\right)^2 - \frac{1}{3!\sqrt{2}}\left(z-\frac{\pi}{4}\right)^3 + \cdots$$

<u>Ex</u> Expand $f(z) = \dfrac{1+2z^2}{z^3+z^5}$

about $z=0$

Note: f is not analytic at $z=0$; cannot have a Taylor/Maclaurin series

$$f(z) = \frac{1+2z^2}{z^3+z^5} = \frac{1}{z^3}\,\frac{1+2z^2}{1+z^2}$$

$$= \frac{1}{z^3}\,\frac{2(1+z^2)-1}{1+z^2}$$

$$= \frac{1}{z^3}\left(2 - \frac{1}{1+z^2}\right)$$

$$= \frac{1}{z^3}\left(2 - \left(1 - z^2 + z^4 - z^6 + \cdots\right)\right)$$

$$= 1/z^3\left(1 + z^2 - z^4 + z^6 - \cdots\right)$$

i 0 $-i$ $\quad |z|<1$

8

University of Idaho

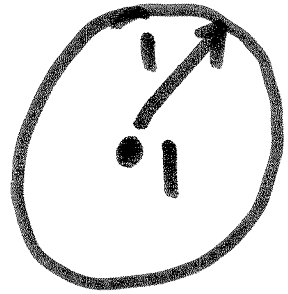
negative powers of z

$$f(z) = \frac{1}{z^3} + \frac{1}{z} - z + z^3 - \cdots$$

$\longrightarrow$ Laurent series for $\frac{1+2z^2}{z^3+z^5}$

Conclusion: Even if f is not analytic at $z = z_0$ it may be possible to find a series rep. for f about z_0 in terms of both negative and positive powers of $z - z_0$. This is the Laurent series of f about z_0

Ex: $f(z) = \dfrac{-1}{(z-1)(z-2)}$



Expand about $z = 1$

$$f(z) = \frac{1}{z-1}\,\frac{1}{2-z} = \frac{1}{(z-1)}\,\frac{1}{1-(z-1)}$$

$$= \frac{1}{(z-1)}\left[1 + (z-1) + (z-1)^2 + \cdots\right]$$

$|z-1| < 1$

$$= \frac{1}{z-1} + 1 + (z-1) + \cdots \longrightarrow \text{Laurent series}$$