MATH 420

COMPLEX VARIABLES

SESSION no. 24

University of Idaho

Singularities

Isolated singularity : $f(z) = \dfrac{1}{z-3}$

$z = 3$ is an isolated singularity.

Non isolated singularity :

$$f(z) = \frac{1}{\sin\left(\frac{\pi}{z}\right)}$$

2

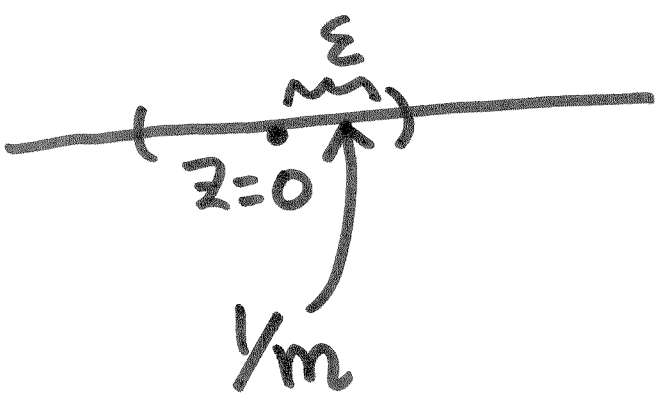
Singularities of $1/\sin(\pi/z)$

$\boxed{z=0} \Rightarrow \sin\left(\frac{\pi}{0}\right)$ undefined

$\boxed{z=\frac{1}{n}}$; $n = \pm 1, \pm 2, \pm 3, \ldots$

$\Rightarrow \sin\left(\frac{\pi}{z}\right) = \sin(n\pi) = 0$

$\Rightarrow f(z) = \frac{1}{0}$: undefined.

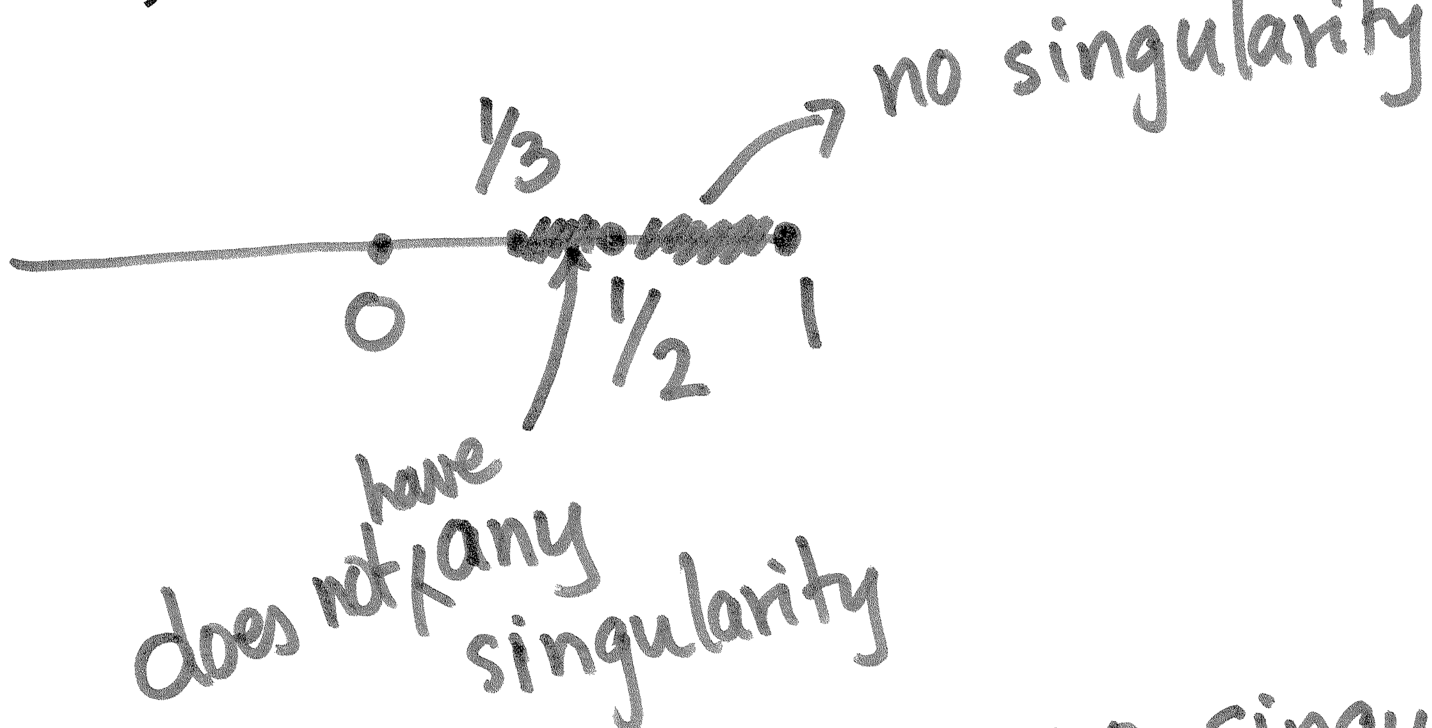


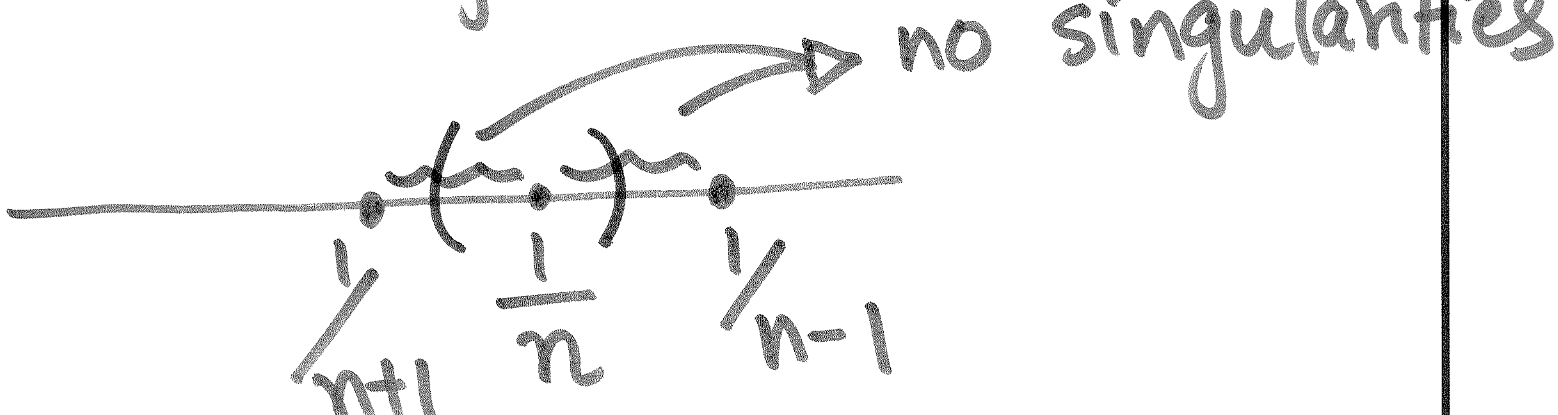
For any ε :

Pick m (an integer):

such that $m > \frac{1}{\varepsilon}$

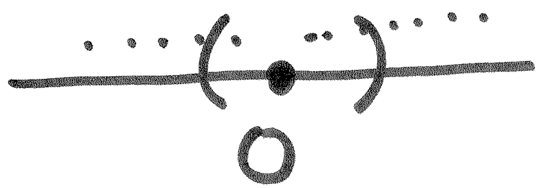
$\Rightarrow \frac{1}{m} < \varepsilon$





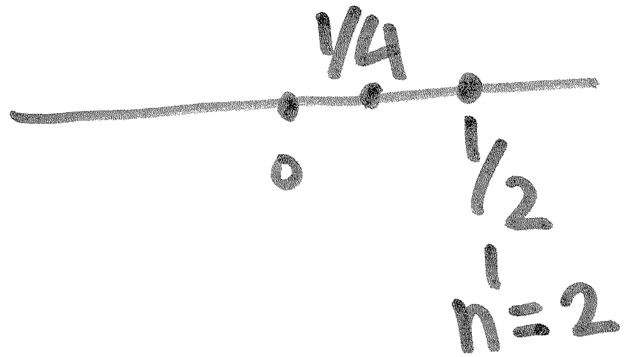
$z = 1/n$ is isolated.

$$\left\{ \frac{1}{n} \right\} \longrightarrow 0$$



$\Rightarrow$ $z = 0$ is not an isolated singularity.

$z = \frac{1}{n}$; $n = \pm 1, \pm 2, \ldots$ are isolated.



Classification of isolated singularities

Let $z = z_0$ be an isolated singularity.

Then about z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}$$

principal part of the Laurent series

Principal part: The part of the Laurent series about z_0 that contains the negative* powers of $z - z_0$.

The principal part is used to classify the isolated singularity:

a) removable b) pole c) essential

7

University of Idaho

a) Removable singularity :

$f(z) = \frac{\sin z}{z}$ $\quad z=0 \Rightarrow \frac{\sin 0}{0}$

undefined

$f(z) = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]$

$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$

principal part $= 0$

$f(z) = \sin z / z$

$$\lim_{z \to 0} \frac{\sin z}{z} = 1$$

Define $f(0) = 1$, then $f(z)$ becomes analytic everywhere $\Rightarrow$ we were able to remove the singularity at $z = 0$.

If the principal part of $f(z)$ about the singularity z_0 is 0 then z_0 is a removable singularity.

10

University of Idaho

b) Pole (of order m)

z_0 is an isolated singularity.

About z_0: $b_m \neq 0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \underbrace{\frac{b_1}{z-z_0} + \cdots + \frac{b_m}{(z-z_0)^m}}_{\text{Has finite \# (m) terms}}$$

for $0 < |z-z_0| < R$

$z = z_0$ is a pole of order m.

Ex: $f(z) = \dfrac{1}{(1-z)^2(2-z)}$

$z=1$: $f(z) = \dfrac{1}{(z-1)^2(1-(z-1))}$

$$= \frac{1}{(z-1)^2}\left[1+(z-1)+(z-1)^2+\cdots\right]$$

$|z-1|<1$

$$= \underbrace{\frac{1}{(z-1)^2} + \frac{1}{(z-1)}}_{\text{principal part}} + 1 + (z-1) + \cdots$$

University of Idaho

$z=1$ is a pole of order 2.

If the principal part has highest negative power of m the $z=z_0$ is a pole of order m.