MATH 420

COMPLEX VARIABLES

SESSION no. 25

Singularities

Isolated

$\frac{1}{z^2}$, $\frac{1}{(z-3)}$

$z=0$ $z=3$

Non-isolated

$\frac{1}{\sin(\pi/z)}$

$z=0$

Removable

$\frac{\sin z}{z}$

$z=0$

Pole

$\frac{1}{(1-z)^2(2-z)}$

$z=1$, $z=2$

Essential

$e^{1/z}$

$z=0$

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$$f(z) = \frac{1}{(1-z)^2(2-z)}$$

$$z=1: \quad \frac{1}{(z-1)^2\left[1-(z-1)\right]} = \underbrace{\frac{1}{(z-1)^2} + \frac{1}{z-1}}_{\text{principal part}} + 1 + \cdots$$

$z=1$ is a pole of order 2.

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Recall: Binomial Thm

$$(a+b)^n = a^n + n a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + \cdots$$

n is positive or negative

Let $a=1$, $b=z-2$, $n=-2$

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$$f = \frac{1}{(1-z)^2(2-z)}$$

About $z=2$: $f(z) = -\frac{1}{(z-2)}\frac{1}{(z-1)^2}$

$$= -\frac{1}{(z-2)}\frac{1}{[1+(z-2)]^2}$$

$$= -\frac{1}{(z-2)}[1+(z-2)]^{-2}$$

BINOMIAL THM.

$$= -\frac{1}{z-2}\left[1-2(z-2)+3(z-2)^2-4(z-2)^3 \cdots\right]$$

$$= -\frac{1}{z-2} + 2 - 3(z-2) \cdots$$

Principal part $= -\frac{1}{z-2}$

$\Rightarrow$ $z=2$ is a pole of order 1

Alternative defn. of a pole:
$z = z_0$ is a pole of order n if

$$\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0$$

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$$f(z) = \frac{1}{(1-z)^2(2-z)}$$

$$\lim_{z \to 1} (-1+z)^2 f(z) = \lim_{z \to 1} (z-1)^2 \frac{1}{(1-z)^2(2-z)}$$

$$= \lim_{z \to 1} \frac{1}{2-z} = 1 \neq 0$$

$\Rightarrow$ $z = 1$ is a pole of order 2

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For $z=2$:

$$\lim_{z\to 2} (z-2)\frac{1}{(1-z)^2(2-z)}$$

$$= \lim_{z\to 2} -\frac{1}{(1-z)^2} = -1 \neq 0$$

$\Rightarrow z=2$ is a pole of order 1.

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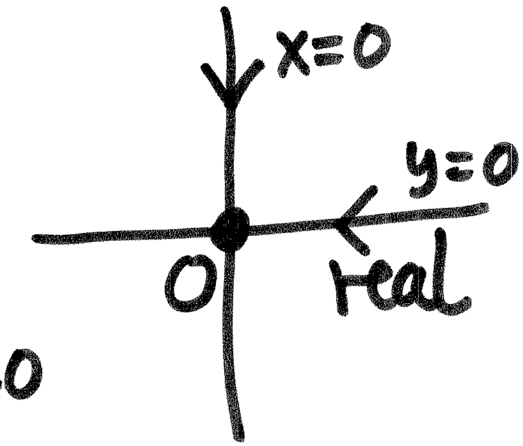
$z = 0$

$$e^{1/z} = 1 + \underbrace{\frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots}_{\text{principal part}}$$

If the principal part has infinitely many terms then the singularity is <u>essential</u>.

$$e^{1/z}$$



Along $y=0$: $z = re^{i\theta}$; $\theta=0$

$z = r$

$e^{1/z} = e^{1/r}$ always real

$\lim\limits_{z\to 0} = \lim\limits_{r\to 0} e^{1/r}$ does not exist

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Along $x=0$: $\theta = \pi/2$, $z = re^{i\pi/2}$

$= ir$

$$e^{1/z} = e^{\frac{1}{ir}} = e^{-i/r}$$

$$\lim_{z \to 0} = \lim_{r \to 0} e^{1/z} = \lim_{r \to 0} e^{-i/r}$$

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Singularities at $z=\infty$: $f(z)$ has a singularity at $z=\infty$ if $f\left(\frac{1}{z}\right)$ has a singularity at $z=0$.

Example: $f(z) = z^2-3z+2$

$$f\left(\frac{1}{z}\right) = \underbrace{\frac{1}{z^2} - \frac{3}{z}}_{\text{principal part}} + 2 \rightarrow \text{Laurent series about } z=0$$

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$\Rightarrow z = 0$ is a pole of order 2 of $f\left(\frac{1}{z}\right)$

$\Rightarrow z = \infty$ is a pole of order 2 of $f(z)$.

$f(z)$