MATH 420

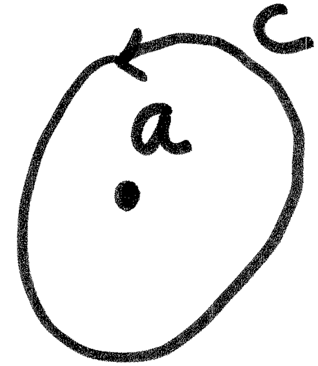
COMPLEX VARIABLES

SESSION no. 26

Calculus of Residues: Residue Thm

$f(z)$ is analytic on and inside C

Let $g(z) = \dfrac{f(z)}{z-a}$



$$\int_C g(z) = \int_C \frac{f(z)}{z-a} = 2\pi i\, f(a)$$

Cauchy's Integral Formula

Note: $g(z)$ has 'a' singularity at $z = a$

2

University of Idaho

What if g has more than one singularity inside C?

The Residue Theorem is useful

a) when there are several singularities within C

b) to evaluale certain real definitē integrals that are otherwise hard to evaluate

3

Residues

Defn: Let z_0 be an isolated singularity of f. About z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$ Laurent series

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

b_1 = coefficient of $\frac{1}{z-z_0}$ is called the residue of f at z_0.

$$b_1 = \operatorname*{Res}_{z=z_0} f(z)$$

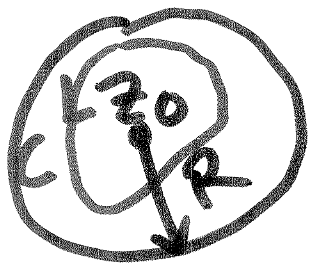
University of Idaho

$f(z)$ is analytic at z_0. About z_0:

Taylor $$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$a_n \underset{\text{Taylor's Thm}}{=} \frac{f^{(n)}(z_0)}{n!} \overset{\text{Int. For.}}{=} \frac{1}{\cancel{n!}}\,\frac{\cancel{n!}}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$\Rightarrow a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$



If f is not analytic at z_0. About z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

Laurent series

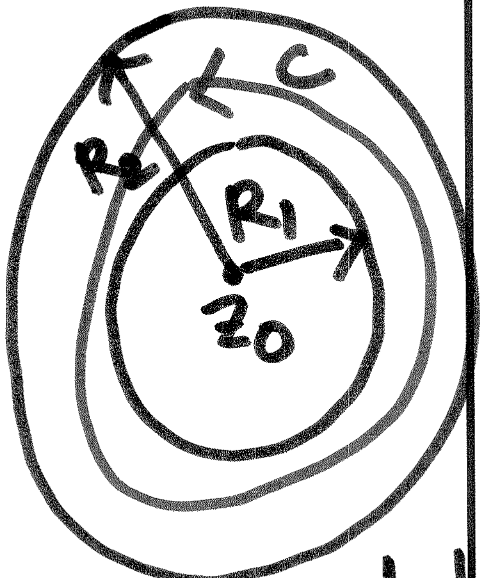
Laurent's Thm

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

$(n=0,1,2,\cdots)$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^{-n+1}}\, dz$$

$(n=1,2,\ldots)$



f is analytic in $R_1 < |z-z_0| < R_2$

University of Idaho

The residue of f at z_0 is

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\,dz$$

$$\Rightarrow \int_C f(z)\,dz = 2\pi i\, b_1$$

$$b_1 = \operatorname*{Res}_{z=z_0} f(z)$$

z_0 : a singularity of f

7

$z=0$: essential singular.

University of Idaho

Ex: Evaluate $\int_C z^2 e^{1/z}$

C: unit circle about origin.

$$f(z) = z^2 e^{1/z} = z^2\left[1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots\right]$$

$$f(z) = z^2 + z + \frac{1}{2!} + \frac{1}{3!\,z} + \cdots$$

$$\operatorname{Res}_{z=0} f = 1/3! = b_1 = \frac{1}{2\pi i}\int_C z^2 e^{1/z}\,dz$$

$$\Rightarrow \int_C z^2 e^{1/z}\, dz = \frac{2\pi i}{3!} = \frac{\pi i}{3}$$

Calculating Residues:

a) Residue at a simple pole (order = 1)

If $z = a$ is a simple pole then

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z \to a} (z-a) f(z)$$

9

University of Idaho

If $z=a$ is a pole of order 1

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a}$$

$$(z-a) f(z) = \sum_{n=0}^{\infty} a_n (z-a)^{n+1} + b_1$$

$$\lim_{z \to a} (z-a) f(z) = b_1 \rightarrow \operatorname*{Res}_{z=a}$$