MATH 420

COMPLEX VARIABLES

SESSION no. 27

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Residue Theorem

z_0 : some isolated singularity of f

About z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

$$b_1 = \operatorname*{Res}_{z=z_0} f(z)$$

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$$b_1 = \frac{1}{2\pi i} \int_C f(z)\,dz$$

$$\Rightarrow \boxed{\int_C f(z)\,dz = 2\pi i\, b_1}$$

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Calculating residues at pole

Thm: If z_0 – a pole of order 1 (simple pole)

then

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z-z_0) f(z)$$

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Ex: $f(z) = \dfrac{3z-1}{z^2-1} = \dfrac{3z-1}{(z-1)(z+1)}$

$z = \pm 1$, poles of order 1.

@ $z=1$:

$\lim_{z\to 1} (z-1)\dfrac{3z-1}{(z-1)(z+1)} = \lim_{z\to 1} \dfrac{3z-1}{z+1} = 1$

$\Rightarrow \operatorname{Res}_{z=1} f(z) = 1.$

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-4/-2

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At $z = -1$:

$$\lim_{z \to -1} (z+1) \frac{(3z-1)}{(z+1)(z-1)} = \lim_{z \to -1} \frac{3z-1}{z-1}$$

$$= 2$$

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Residue at a pole of order m:

If $z = z_0$ is a pole of order m then the residue at z_0 is

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f\right]$$

$$= \frac{1}{(m-1)!} \cdot \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

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$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \cdots + \frac{b_m}{(z-z_0)^m}$$

$b_1 \rightarrow$ Res

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} + b_m + \cdots + b_1 (z-z_0)^{m-1}$$

RHS = Taylor series about z_0 of $(z-z_0)^m f(z)$.

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$$\Rightarrow b_1 = \frac{1}{(m-1)!} \frac{d^{m-1}}{d\cancel{m}z^{m-1}} \left[(z-z_0)^m f(z) \right]$$

$$\Rightarrow b_1 = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$$

$$\operatorname*{Res}_{z=z_0} f$$

Example: $f(z) = \dfrac{\cos z}{z^6}$

$z = 0$ (singularity, pole of order 6)

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{5!} \lim_{z \to 0} \frac{d^5}{dz^5}\left[z^6 \frac{\cos z}{z^6} \right]$$

$$= \frac{1}{5!} \lim_{z \to 0} \frac{d^5}{dz^5} \cos z$$

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$$= \frac{1}{5!} \lim_{z \to 0} (-\sin z) = 0$$

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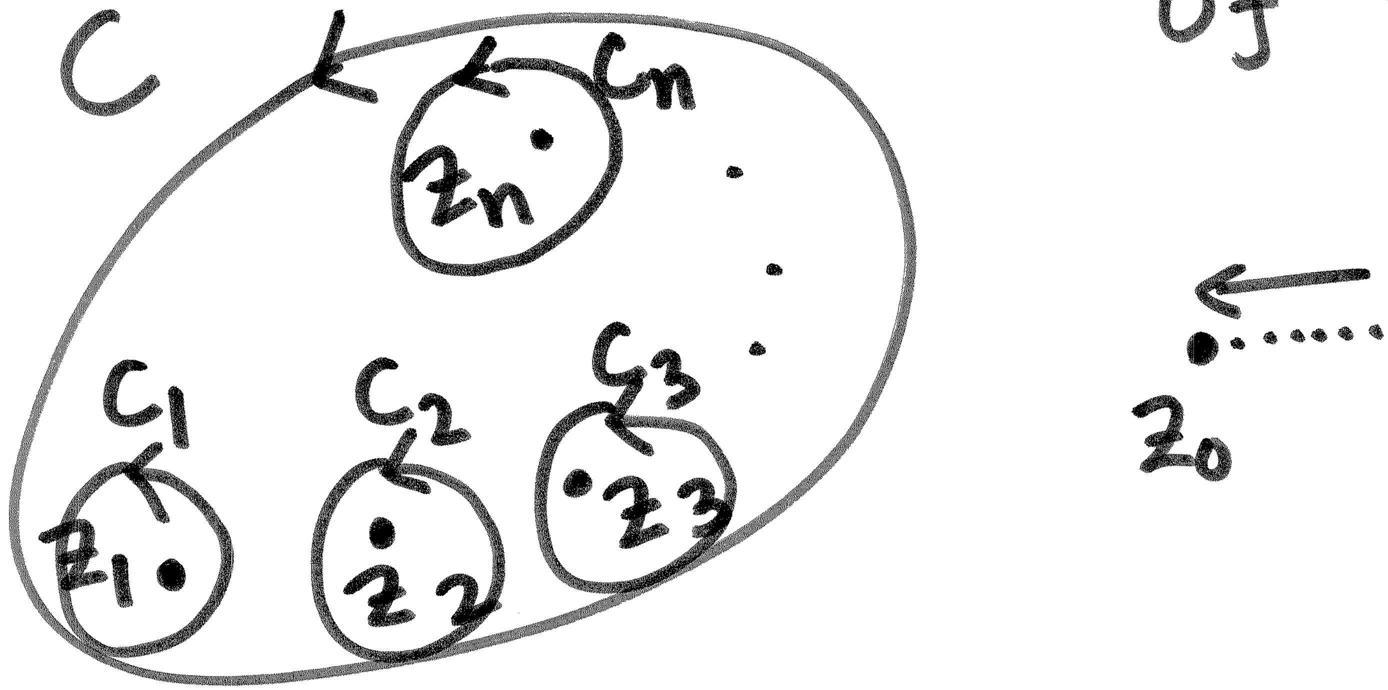
Residue Theorem

Compute $\int_C f(z)$; $z_1, \dots, z_n$: singularities of f.



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Thm: If f is analytic everywhere on and inside C except at isolated singularities $z_1, \cdots, z_n$ then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k}(f)$$

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Proof:

$$\int_C f = \int_{C_1} f + \int_{C_2} f + \cdots + \int_{C_n} f$$

$$= 2\pi i \operatorname*{Res}_{z=z_1} + 2\pi i \operatorname*{Res}_{z=z_2} + \cdots$$

$$+ 2\pi i \operatorname*{Res}_{z=z_n}$$

$$= 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k}$$

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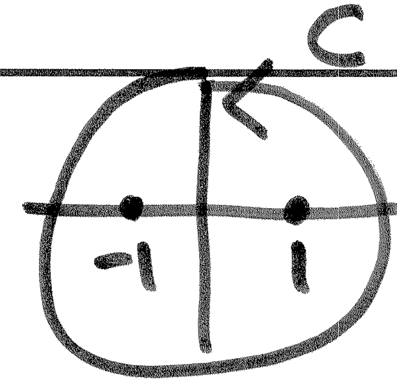
Evaluate $\displaystyle\int_C \frac{3z-1}{z^2-1}\,dz$



where C contains both ± 1.

$\operatorname*{Res}_{z=1} f = 1$; $\operatorname*{Res}_{z=-1} f = 2$.

By the Residue Thm

$$\int_C \frac{3z-1}{z^2-1} = 2\pi i\,(1+2) = 6\pi i$$