MATH 420

COMPLEX VARIABLES

SESSION no. 28

University of Idaho

Midterm Review

8 a) $\dfrac{e^z - 1}{z}$ $\quad z = 0$ a singularity

$$\frac{1}{z}\left[\cancel{1} + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots - \cancel{1}\right]$$

$$= 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$$

Principal part $= 0 \Rightarrow z = 0$ is removable

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Let $f(z) = \dfrac{e^z - 1}{z}$

$$f\left(\frac{1}{z}\right) = z\left(e^{1/z} - 1\right)$$

$$= z\left[\cancel{1} + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots \cancel{-1}\right]$$

$$= 1 + \frac{1}{2!\,z} + \frac{1}{3!\,z^2} + \cdots$$

$z = 0$ is a singularity of $f\left(\frac{1}{z}\right)$ (essential)

$\Rightarrow z = \infty$ is an ess. sing. of $f(z)$.

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8b) $f = \dfrac{\cos 2z}{z^2}$ $\quad z=0$ is a singularity.

$$\frac{\cos 2z}{z^2} = \frac{1}{z^2}\left[1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} - \cdots\right]$$

$$= \frac{1}{z^2} - \frac{4}{2!} + \frac{16z^2}{4!} - \cdots$$

Principal part $= \dfrac{1}{z^2} + 0 \cdot \dfrac{1}{z}$

$z=0$ is a pole of order 2

Alternatively :

$$\lim_{z\to 0} z^2 f(z) = \lim_{z\to 0} z^2 \frac{\cos 2z}{z^2}$$

$$= \lim_{z\to 0} \cos 2z = 1 \neq 0$$

$\Rightarrow$ $z = 0$ is a pole of order 2.

Check @ $z = \infty$.

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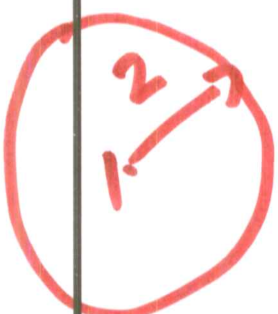
7. Expand $\frac{1}{z^2-1}$ about $z=1$.

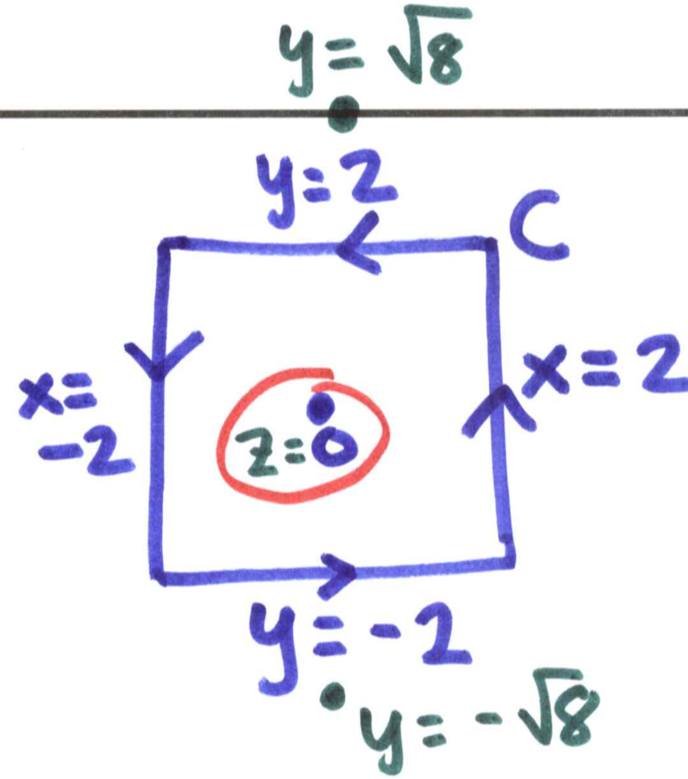
$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{z-1}\,\frac{1}{2+(z-1)}$$



$$= \frac{1}{z-1}\,\frac{1}{2\left(1+\frac{z-1}{2}\right)}\ ;\quad \left|\frac{z-1}{2}\right|<1 \Rightarrow |z-1|<2$$

$$= \frac{1}{2(z-1)}\left[1-\frac{z-1}{2}+\frac{(z-1)^2}{4}-\dots\right]$$

3 ~~a~~ b) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}$



$$\frac{\cos z}{z(z^2+8)} = \frac{\cos z}{z(z+i\sqrt{8})(z-i\sqrt{8})}$$

$z=0$ - inside C

$z = \pm i\sqrt{8}$ lie outside C

$f = \dfrac{\cos z}{z^2+8}$ is analytic on & inside C

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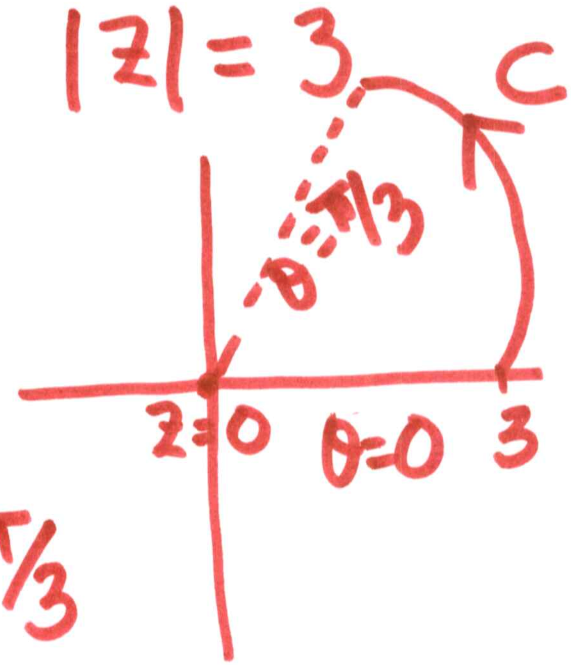
Int. For.

$$\int_C \frac{\cos z}{z(z^2+8)} = \int_C \frac{f(z)}{z-0} \overset{\downarrow}{=} 2\pi i f(0)$$

$$= 2\pi i \frac{1}{8}$$

$$f(0) = \frac{\cos 0}{8} = \frac{1}{8} \qquad = \frac{\pi}{4} i$$

#1. $\int_C \frac{z+2}{3z}\,dz$



$$C:\ z = 3e^{i\theta}\,;\ 0 \le \theta \le \pi/3$$

$$dz = 3ie^{i\theta}\,d\theta$$

$$\int_0^{\pi/3} \frac{3e^{i\theta}+2}{3\cdot\cancel{3}\cancel{e^{i\theta}}}\,\cancel{3}i\cancel{e^{i\theta}}\,d\theta$$

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$$i\int_0^{\pi/3} \frac{3e^{i\theta}+2}{3}\,d\theta$$

$$= i\int_0^{\pi/3}\left[e^{i\theta}+\frac{2}{3}\right]d\theta$$

$$= \cancel{i}\,\frac{e^{i\theta}}{\cancel{i}}\Big|_0^{\pi/3} + i\,\frac{2}{3}\theta\Big|_0^{\pi/3}$$

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$$e^{i\pi/3} - 1 + i\frac{2}{3}\frac{\pi}{3}$$