MATH 420

COMPLEX VARIABLES

SESSION no. 29

z_0 : an isolated singularity of f

Expand f about z_0 :

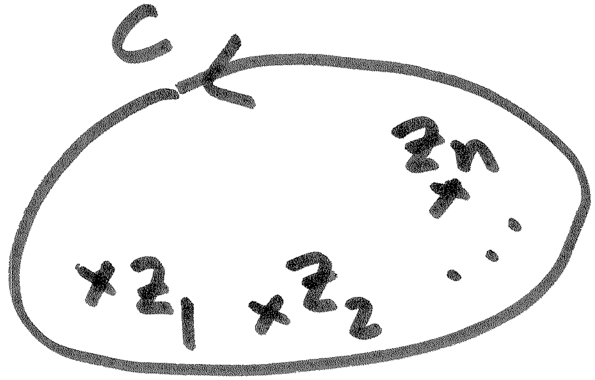
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

$$b_1 = \operatorname*{Res}_{z=z_0} f(z) \ ; \ b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz$$

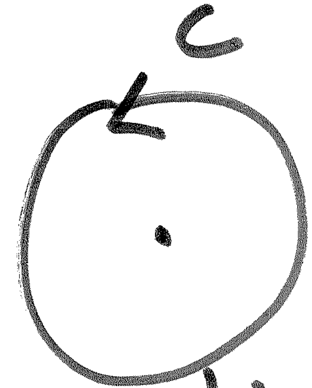
$$\left[\int_C f(z) = 2\pi i\, b_1 = 2\pi i \operatorname*{Res}_{z=z_0} f(z) \right]$$

Residue Thm: If f is analytic everywhere on and inside C except at $z_1, z_2, \dots, z_n$ then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{\operatorname{Res}} f(z)$$



Ex: Evaluate $\int_C e^{1/z^2}$



C: circle $|z| = 1$

$z = 0$ is a singularity (essential)

$$e^{1/z^2} = 1 + {}^{0\cdot\frac{1}{z}+}_{\wedge}\frac{1}{z^2} + \frac{1}{2!\,z^4} + \frac{1}{3!\,z^6} + \cdots$$

$$\operatorname*{Res}_{z=0} e^{1/z^2} = 0 \Rightarrow \int_C e^{1/z^2}\,dz = 2\pi i \cdot 0$$

$$= 0$$

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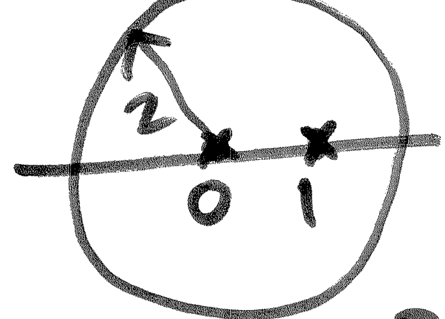
Evaluate $\displaystyle\int_C \frac{5z-2}{z(z-1)}\,dz$



C: the circle $|z| = 2$

Singularities are @ $z = \overbrace{0, 1}^{\text{pole; order } 1}$

$$\operatorname*{Res}_{z=0} = \lim_{z \to 0} z \, \frac{5z-2}{z(z-1)} = 2 \neq 0$$

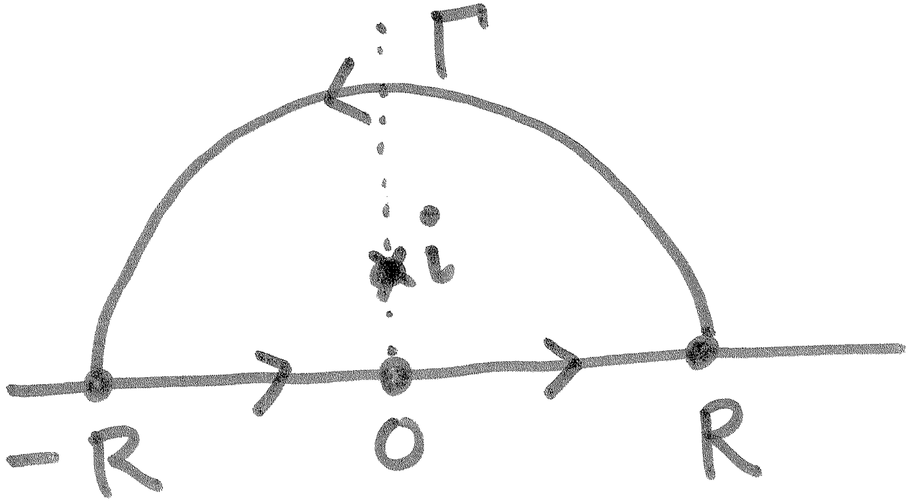
$$\operatorname*{Res}_{z=1} = \lim_{z \to 1} (z-1) \, \frac{5z-2}{z(z-1)} = 3 \neq 0$$

$$\int_C \frac{5z-2}{z(z-1)}\,dz = 2\pi i(2+3) = 10\pi i$$

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$

From calculus: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \tan^{-1} x \Big|_{-\infty}^{\infty}$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$



$C : \Gamma \cup [-R, R]$

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both simple poles

Let $f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$; singularities at $z = \pm i$

Only $z = i$ is inside C

$$\int_C \frac{1}{z^2+1} = 2\pi i \operatorname*{Res}_{z=i} f$$

$$= 2\pi i \lim_{z \to i} (z-i) \frac{1}{(z-i)(z+i)}$$

Residue Thm

$$= \cancel{2}\pi \cancel{i} \frac{1}{\cancel{2}\cancel{i}} = \pi$$

$$\pi = \int_C \frac{dz}{z^2+1} = \int_\Gamma \frac{dz}{z^2+1} + \int_{-R}^{R} \frac{dx}{x^2+1}$$

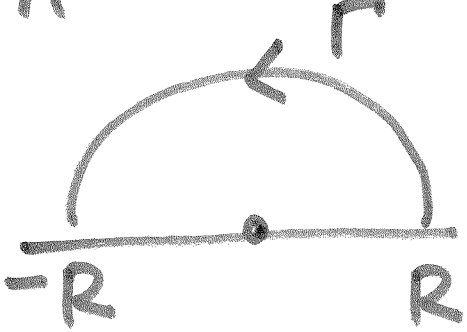
$$\pi = \lim_{R\to\infty} \int_\Gamma \frac{dz}{z^2+1} + \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1}$$

$$\Rightarrow \pi = \underbrace{\lim_{R\to\infty} \int_\Gamma \frac{dz}{z^2+1}}_{\to 0} + \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

$$0 \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \boxed{\pi}$$

$$\Gamma : z = Re^{i\theta} \; ; \; 0 \le \theta \le \pi$$

$$dz = Rie^{i\theta} d\theta$$



$$\lim_{R\to\infty} \int_{\Gamma} \frac{dz}{z^2+1} = \lim_{R\to\infty} \int_0^{\pi} \frac{Rie^{i\theta} d\theta}{R^2 e^{2i\theta}+1} = 0$$

<u>Try</u>:

Integrale: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^6+1}$