

MATH 420

COMPLEX VARIABLES

SESSION no. 29

z_0 : an isolated singularity of f

Expand f about z_0 :

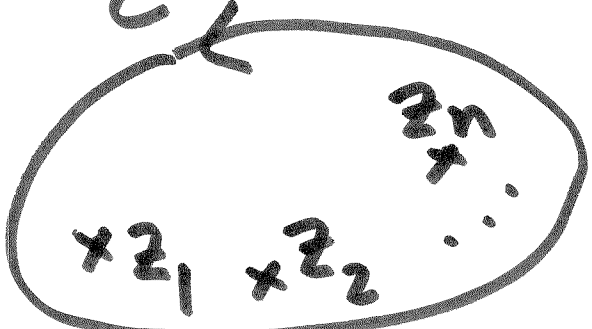


$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots$$

$$b_1 = \operatorname{Res}_{z=z_0} f(z); \quad b_1 = \frac{1}{2\pi i} \int_C f(z) dz$$

$$\left[\int_C f(z) = 2\pi i b_1 = 2\pi i \operatorname{Res}_{z=z_0} f(z) \right]$$

Residue Thm: If f is analytic everywhere on and inside C except at z_1, z_2, \dots, z_n then

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \operatorname{Res}_{z=z_k} f(z)$$


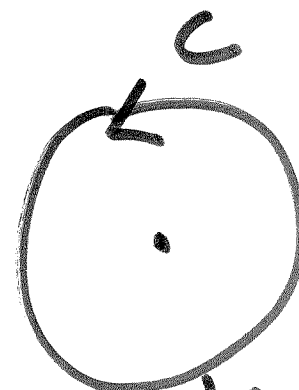
The diagram shows a closed contour C in the complex plane. Inside the contour, there are several points marked with 'x' and labeled z_1, z_2, \dots, z_n , representing poles of the function $f(z)$. The contour C is drawn as an oval shape, with an arrow indicating a counter-clockwise direction of integration.

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Ex: Evaluate $\int_C e^{1/z^2}$

C : circle $|z|=1$



$z=0$ is a singularity (essential)

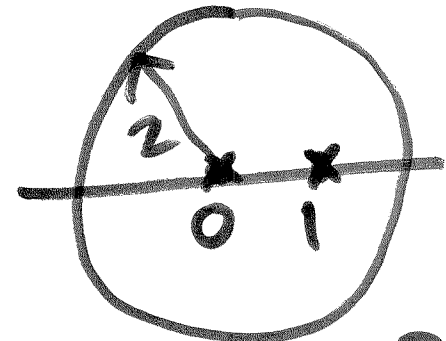
$$e^{1/z^2} = 1 + \frac{0 \cdot \frac{1}{z} + 1}{1! z^2} + \frac{1}{2! z^4} + \frac{1}{3! z^6} + \dots$$

$$\text{Res}_{z=0} e^{1/z^2} = 0 \Rightarrow \int_C e^{1/z^2} dz = 2\pi i \cdot 0 = 0$$

Evaluate $\int_C \frac{5z-2}{z(z-1)} dz$

C : the circle $|z|=2$

Singularities are @ $z=0, 1$ pole; order 1



$$\text{Res}_{z=0} = \lim_{z \rightarrow 0} z \frac{5z-2}{z(z-1)} = 2 \neq 0$$

$$\text{Res}_{z=1} = \lim_{z \rightarrow 1} (z-1) \frac{5z-2}{z(z-1)} = 3 \neq 0$$

$$\int_C \frac{5z-2}{z(z-1)} dz = 2\pi i (2+3) = 10\pi i$$

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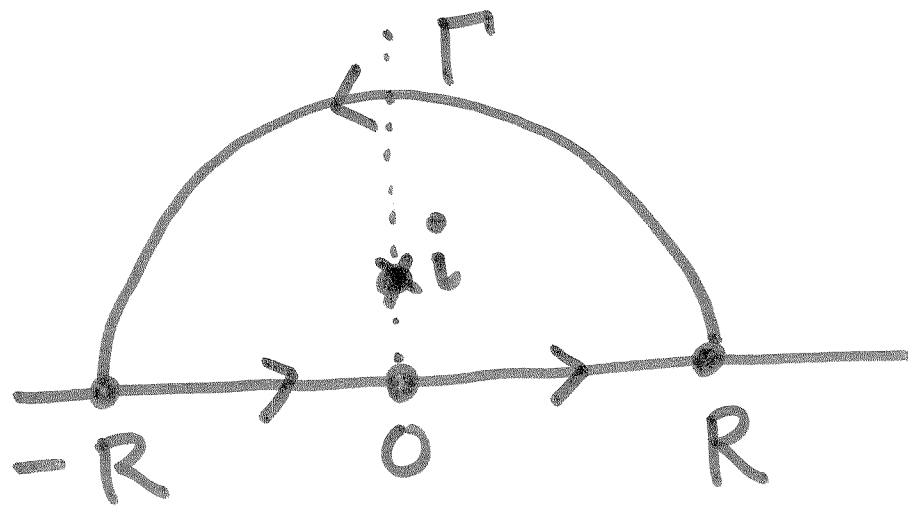
Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

From calculus:

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \tan^{-1} x \Big|_{-\infty}^{\infty}$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$



$$C: \Gamma \cup [-R, R]$$

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both simple poles

Let $f(z) = \frac{1}{z^2 + 1} = \frac{1}{(z-i)(z+i)}$; singularities at $z = \pm i$

Only $z = i$ is inside C

$$\int_C \frac{1}{z^2 + 1} = 2\pi i \operatorname{Res}_{z=i} f$$

$$= 2\pi i \lim_{z \rightarrow i} (z-i) \frac{1}{(z-i)(z+i)}$$

Residue
Thm

$$= 2\pi i \frac{1}{2i} = \pi$$

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$$\pi = \int_C \frac{dz}{z^2+1} = \int_{\Gamma} \frac{dz}{z^2+1} + \int_{-R}^R \frac{dx}{x^2+1}$$

$$\pi = \lim_{R \rightarrow \infty} \int_{\Gamma} \frac{dz}{z^2+1} + \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{x^2+1}$$

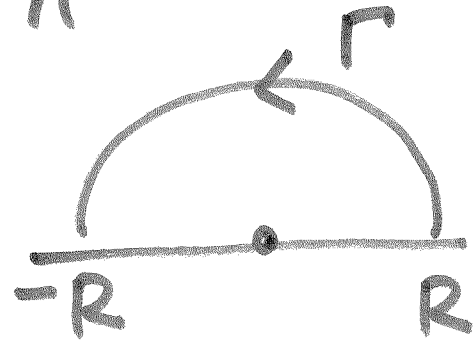
$$\Rightarrow \pi = \lim_{R \rightarrow \infty} \int_{\Gamma} \frac{dz}{z^2+1} + \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

$$\circ \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \boxed{\pi}$$

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$$\Gamma: z = R e^{i\theta} ; 0 \leq \theta \leq \pi$$

$$dz = R i e^{i\theta} d\theta$$



$$\lim_{R \rightarrow \infty} \int_{\Gamma} \frac{dz}{z^2 + 1} = \int_0^{\pi} \frac{R i e^{i\theta} d\theta}{R^2 e^{2i\theta} + 1} = 0$$

Try:
Integrate: $\int_{-\infty}^{\infty} \frac{dx}{x^6 + 1}$