MATH 420

COMPLEX VARIABLES

SESSION no. 30

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Application of the Residue Thm.

in

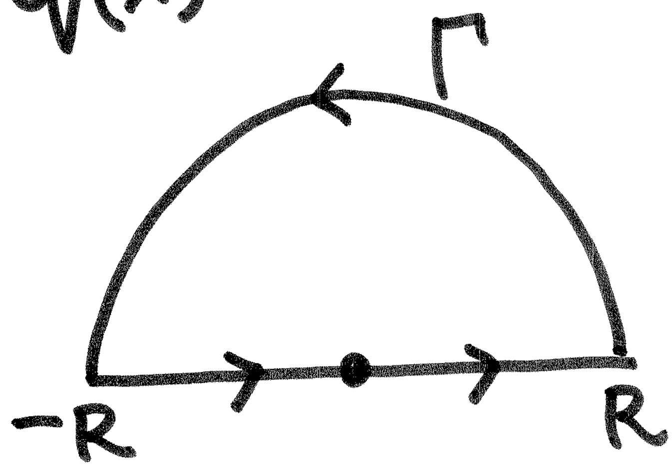
evaluating real integrals

The following types of real integrals are most common:

1. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$, $f(x)$ is a rational function

$$f(x) = \frac{p(x)}{q(x)}$$

Technique: $\displaystyle\int_{C} f(z)\,dz$



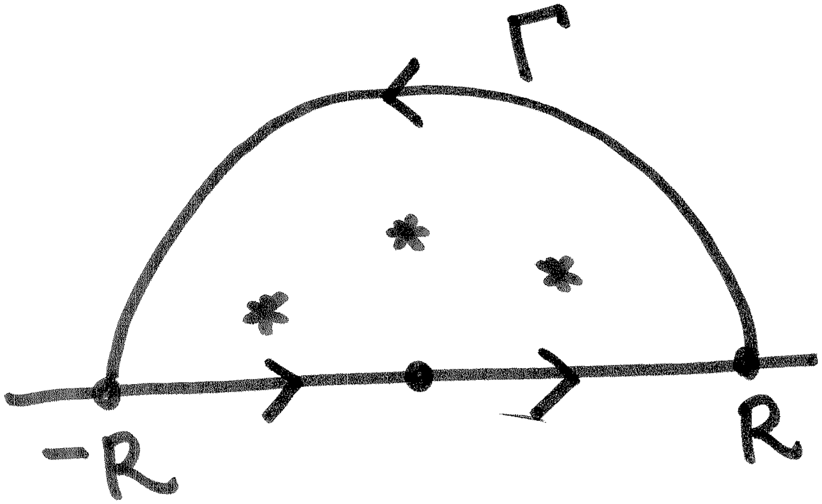
$C = \Gamma \cup [-R, R]$

Ex: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ (last class)

Ex: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^6}$

$f(x) = \dfrac{1}{1+x^6}$

$f(z) = \dfrac{1}{1+z^6}$



$C = \Gamma \cup [-R, R]$

$$\int_C \frac{1}{z^6+1}\,dz = \int_\Gamma + \int_{-R}^{R}$$

Diversion: Another result of calculating residues at simple poles

Theorem: Let $f(z) = \dfrac{p(z)}{q(z)}$.

The zeros of $q(z)$ are the poles of $f(z)$. If z_0 is a simple pole:

$$\left[\ \operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}\ \right]$$

$\int_C \frac{dz}{1+z^6}$ – First find the zeros of z^6+1

(simple) poles of $\frac{1}{1+z^6}$ i.e. Solve $z^6+1=0$

$\Rightarrow z^6 = -1$

$\Rightarrow z^6 = e^{i\pi + i2n\pi}$

$\Rightarrow z = e^{i(2n+1)\pi/6}$ $n = 0, 1, \ldots, 5$

$\Rightarrow z = \underset{*}{e^{i\pi/6}}, \underset{*}{e^{i\pi/2}}, \underset{*}{e^{i5\pi/6}}, e^{i7\pi/6}, e^{i9\pi/6}, e^{i11\pi/6}$

$$\int_C \frac{dz}{1+z^6} = 2\pi i\left[\operatorname*{Res}_{z=e^{i\pi/6}} + \operatorname*{Res}_{z=e^{i\pi/2}} + \operatorname*{Res}_{z=e^{i5\pi/6}}\right]$$

$$\operatorname*{Res}_{z=e^{i\pi/6}} \frac{\overbrace{1}^{p}}{\underbrace{1+z^6}_{q}} = \left.\frac{1}{6z^5}\right|_{z=e^{i\pi/6}} = \frac{1}{6}e^{-i5\pi/6}$$

$$\operatorname*{Res}_{z=e^{i\pi/2}} \frac{1}{1+z^6} = \frac{1}{6}e^{-i5\pi/2}; \quad \operatorname*{Res}_{z=e^{i5\pi/6}} = \frac{1}{6}e^{-i25\pi/6}$$

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$$\int_C \frac{dz}{1+z^6} \overset{\text{Residue Thm}}{=} 2\pi i\left[\frac{1}{6}e^{-i5\pi/6} + \frac{1}{6}e^{-i\frac{5\pi}{2}} + \frac{1}{6}e^{-i\frac{25\pi}{6}}\right]$$

check

$$= \boxed{2\pi/3}$$

$$\int_C \frac{dz}{1+z^6} = \int_\Gamma \frac{dz}{1+z^6} + \int_{-R}^{R} \frac{dx}{1+x^6}$$

$$\Rightarrow\ 2\pi/3 = \lim_{R\to\infty}\left[\underbrace{\cancel{\int_\Gamma \frac{dz}{1+z^6}}}_{0} + \int_{-\infty}^{\infty} \frac{dx}{1+x^6}\right]$$

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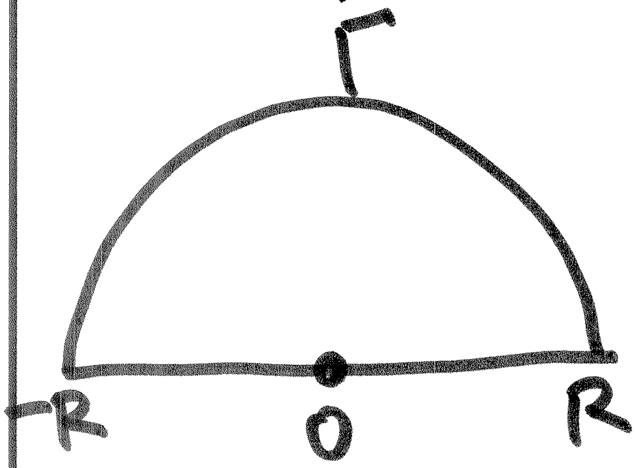

$$\Gamma : z = Re^{i\theta} ; 0 \le \theta \le \pi$$

$$dz = iRe^{i\theta} d\theta$$

$$\lim_{R\to\infty} \int_\Gamma \frac{dz}{1+z^6} = \lim_{R\to\infty} \int_0^\pi \frac{iRe^{i\theta} d\theta}{1+R^6 e^{i6\theta}}$$

$$= 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{1+x^6} = \frac{2\pi}{3}.$$

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If f is even $\left(f(x) = f(-x) \right)$

Then $\displaystyle\int_0^\infty f(x)\,dx = \frac{1}{2}\int_{-\infty}^\infty f(x)\,dx$

$$\int_0^\infty \frac{dx}{1+x^6} = \frac{1}{2}\ \frac{2\pi}{3} = \frac{\pi}{3}.$$