# MATH 420

# COMPLEX VARIABLES

SESSION no. 31

# Application of the Residue Thm.

# in

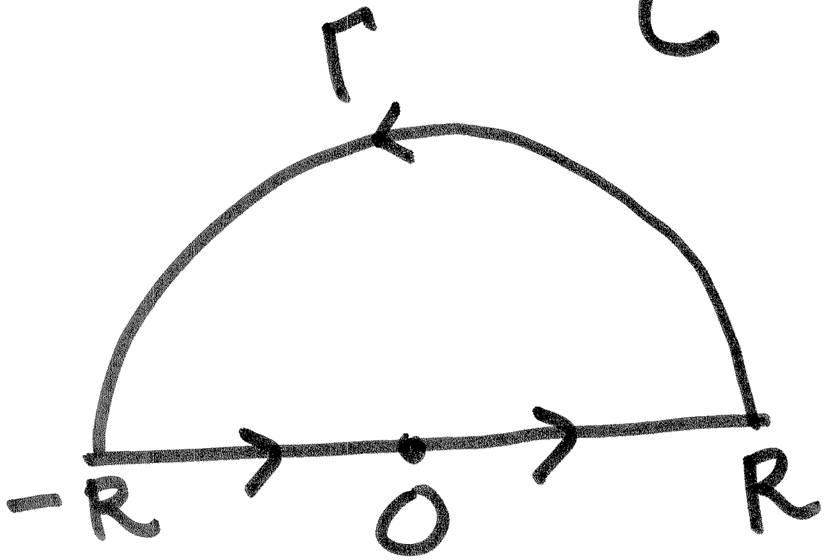
# evaluating real integrals

1

University of Idaho

1. $$\int_{-\infty}^{\infty} f(x)\,dx \quad ; \quad f(x) = \frac{p(x)}{q(x)}$$

Evaluate $$\int_C f(z)\,dz$$




$$C = \Gamma \cup [-R, R]$$

2. $$\int_0^{2\pi} G(\sin\theta, \cos\theta)\, d\theta$$


$$0 \le \theta \le 2\pi \longleftrightarrow \underbrace{z = e^{i\theta}}_{\to\, dz = ie^{i\theta}\, d\theta}; \quad z \in C.$$

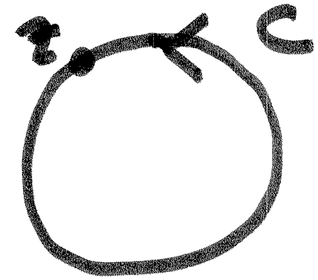
$$e^{i\theta} = \cos\theta + i\sin\theta \quad \Rightarrow d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

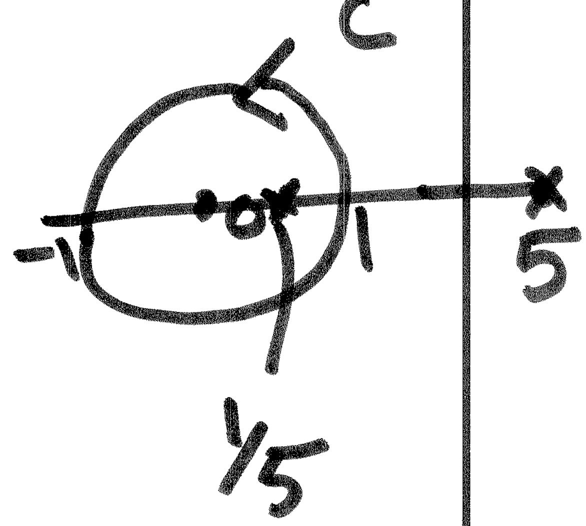
Ex: Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{13 - 5\cos\theta}$

$z = e^{i\theta}$, $0 \le \theta \le 2\pi$



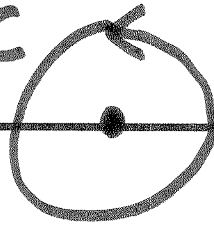

$dz = i e^{i\theta}\, d\theta = i z\, d\theta$

$\Rightarrow \quad d\theta = \dfrac{dz}{iz}$ ✓




$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ ✓

$z = e^{i\theta}$ unit circ. C



$$\int_0^{2\pi} \frac{d\theta}{13-5\cos\theta} = \int_C \frac{dz}{iz\left(13-\frac{5}{2}\left(z+\frac{1}{z}\right)\right)}$$

$$\frac{1}{13-\frac{5}{2}\left(z+\frac{1}{z}\right)} = \frac{-2z}{5z^2-26z+5}$$

$$= \frac{-2z}{(5z-1)(z-5)}$$

5



$$\int_0^{2\pi} \frac{d\theta}{13 - 5\cos\theta} = -\int_C \frac{2\cancel{z}\,dz}{i\cancel{z}(5z-1)(z-5)}$$

$$= -\frac{2}{i}\int_C \frac{dz}{(5z-1)(z-5)}$$

$$f(z) = \frac{1}{(5z-1)(z-5)}$$ ; Singularities

@ $z = \frac{1}{5}, 5$ - only $z = 1/5$ is inside C

simple pole

$$\int_0^{2\pi} \frac{d\theta}{13 - 5\cos\theta} = -\frac{2}{i} \int_C \underbrace{\frac{dz}{(5z-1)(z-5)}}_{\to f}$$

Residue Thm $\to$

$$= -\frac{2}{i}\, 2\pi i \operatorname*{Res}_{z=1/5} f(z)$$

$$\operatorname*{Res}_{z=1/5} f(z) = \lim_{z \to \frac{1}{5}} \left(z - \frac{1}{5}\right) \frac{1}{(5z-1)(z-5)}$$

$$= \lim_{z \to \frac{1}{5}} \cancel{\left(z - \frac{1}{5}\right)} \frac{1}{5\cancel{\left(z - \frac{1}{5}\right)}(z-5)}$$

$$= -\frac{1}{24}$$

7

University of Idaho

$$\int_0^{2\pi} \frac{d\theta}{13-5\cos\theta} = -\frac{\cancel{2}}{\cancel{i}}\,\cancel{2}\pi \cancel{i}\left(-\frac{1}{\cancel{24}_{6}}\right)$$

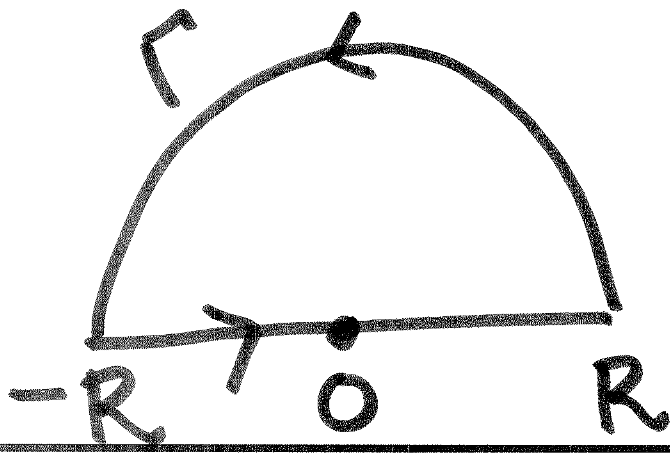
$$= \pi/6$$

University of Idaho

3. $\int_{-\infty}^{\infty} f(x)\cos mx \, dx$ , $\int_{-\infty}^{\infty} f(x)\sin mx \, dx$

$f(x)$ - rational function

$m$ - positive constant




$$\int_C f(z)\, e^{imz}\, dz$$

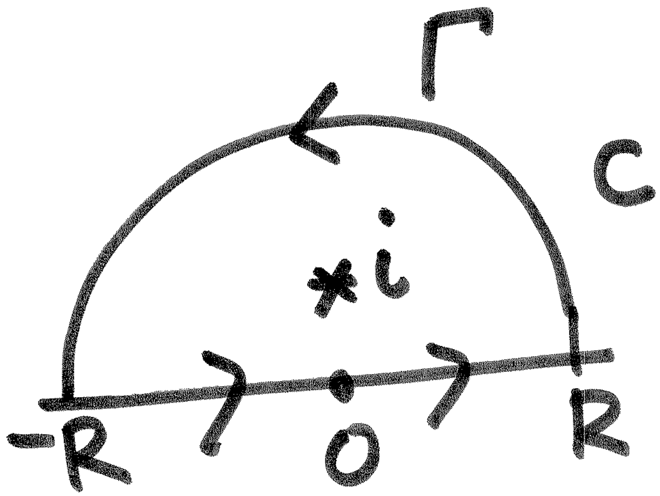
$C = \Gamma \cup [-R, R]$

$e^{imz} = \cos mz + i \sin mz$

Evaluate $\displaystyle\int_0^{\infty} \frac{\cos mx}{1+x^2}$, $m>0$

Let

$$f(z) = \frac{e^{imz}}{1+z^2}.$$




$$\int_C \frac{e^{imz}}{1+z^2}\,dz \qquad C = \Gamma \cup [-R, R]$$

$$\frac{e^{imz}}{1+z^2} = \frac{e^{imz}}{(z-i)(z+i)}$$

University of Idaho

$f(z)$ has singularities @ $\pm i$

Only $i \in C$.

Residue Thm

$$\int_C \frac{e^{imz}}{1+z^2} = 2\pi i \operatorname*{Res}_{z=i} f(z)$$

$\rightarrow f$

$$\operatorname*{Res}_{z=i} = \lim_{z \to i} \cancel{(z-i)} \frac{e^{imz}}{\cancel{(z-i)}(z+i)} = \frac{e^{-m}}{2i}$$

11

$$e^{imz} = \cos mz + i \sin mz$$

University of Idaho

$$\int_C \frac{e^{imz}}{1+z^2}\,dz = 2\pi i \,\frac{e^{-m}}{2i} = \pi e^{-m}$$

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\cos mx}{1+x^2}\,dx + i\int_{-R}^{R}\frac{\sin mx}{1+x^2}\,dx + \cancel{\int_\Gamma \frac{e^{imz}}{1+z^2}\,dz} \to 0$$

$$= \pi e^{-m}$$

$$\frac{e^{imz}}{1+z^2} \xrightarrow[z^2 = R^2 e^{i2\theta}]{\text{on } \Gamma \;\; z = Re^{i\theta}} \frac{e^{imz}}{1+R^2 e^{i2\theta}} \xrightarrow[R\to\infty]{} 0$$

12

University of Idaho

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx}{1+x^2}\,dx + i\int_{-\infty}^{\infty} \frac{\sin mx}{1+x^2}\,dx$$

$$= \pi e^{-m}$$

Equating real & img. parts

$$\int_{-\infty}^{\infty} \frac{\sin mx}{1+x^2}\,dx = 0$$

$$\int_{-\infty}^{\infty} \frac{\cos mx}{1+x^2}\,dx = \pi e^{-m} \xRightarrow{\text{by evenness}} \int_{0}^{\infty} \frac{\cos mx}{1+x^2} = \frac{\pi}{2} e^{-m}.$$