MATH 420

COMPLEX VARIABLES

SESSION no. 32

University of Idaho

Definite Integrals - Residue Thm

$$\int_{-\infty}^{\infty} f(x)\,dx, \quad \int_{0}^{2\pi} G(\sin\theta, \cos\theta)\,d\theta, \quad \int_{-\infty}^{\infty} f(x)\begin{Bmatrix}\sin mx\\ \cos mx\end{Bmatrix},$$

$m > 0$

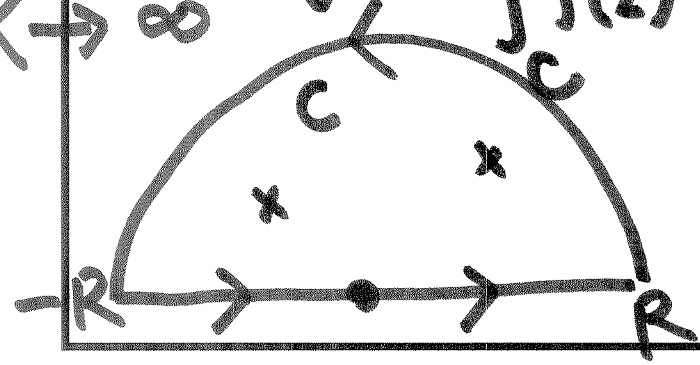
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6}, \quad \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

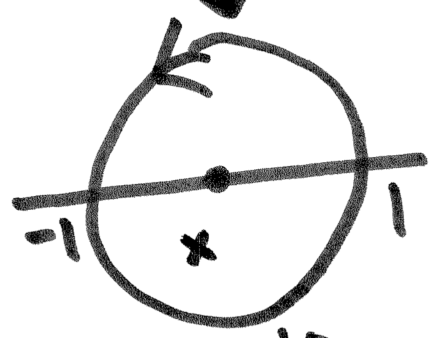
$$\int_{0}^{2\pi} \frac{d\theta}{13-5\cos\theta}$$

$$\int_{0}^{\infty} \frac{\cos mx}{1+x^2}\,dx$$

$R \to \infty$

$\int_C f(z)$

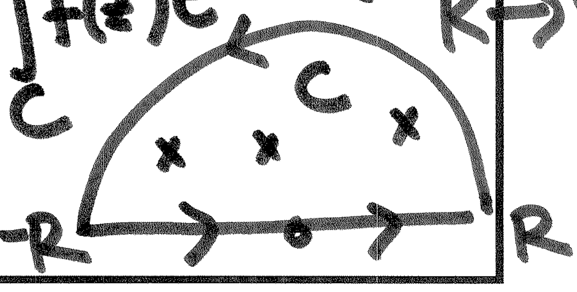




$z = e^{i\theta}$

$\int_C f(z)e^{imz}$

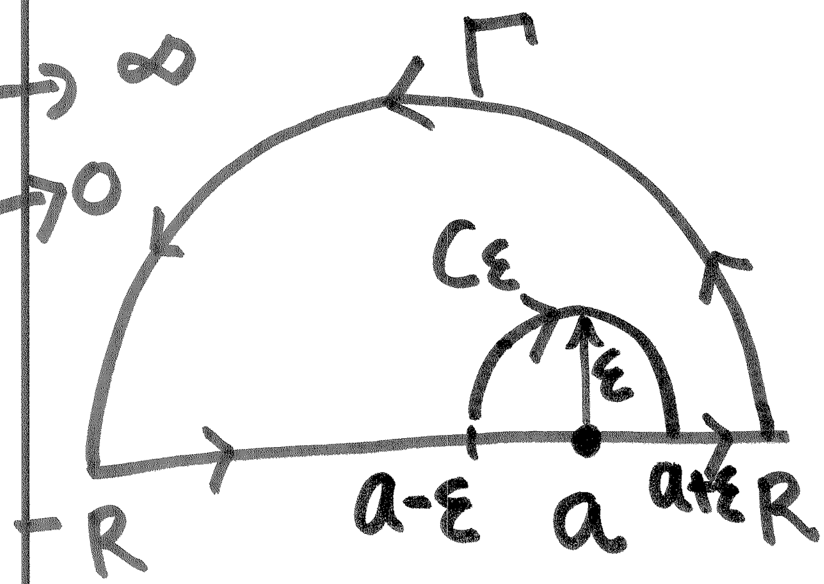
$R \to \infty$



Indenting the contour:

$R \to \infty$

$\varepsilon \to 0$



In $\int_{-\infty}^{\infty} f(x)$ or $\int_{-\infty}^{\infty} f(x)\begin{Bmatrix}\cos mx \\ \sin mx\end{Bmatrix}$:

what if f has a singularity on the real axis?

$f(a) = \infty$, a is singularity.

$\int_C$: $C = \Gamma \cup [-R, a-\varepsilon] \cup C_\varepsilon \cup [a+\varepsilon, R]$

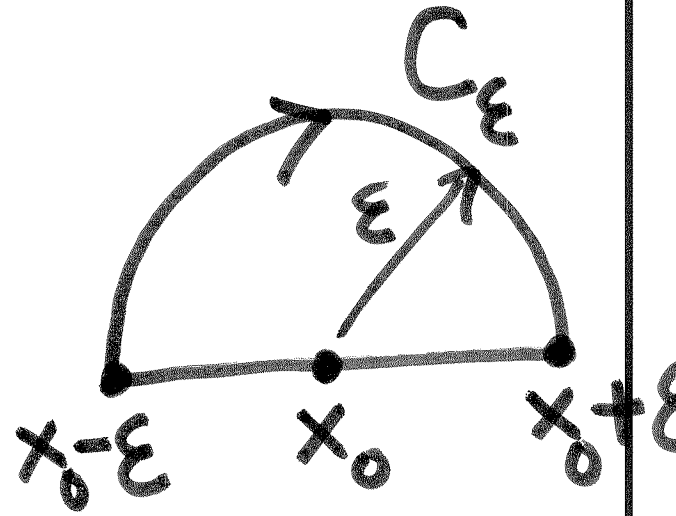
$R \to \infty$, $\varepsilon \to 0$

C_ε – semicircle, center @ a, rad $= \varepsilon$

Theorem: $f(z)$ has a simple pole at $z = x_0$ (on the real line); residue of f at $x_0 = B_0$.

Then

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z)\,dz = -B_0 \pi i$$



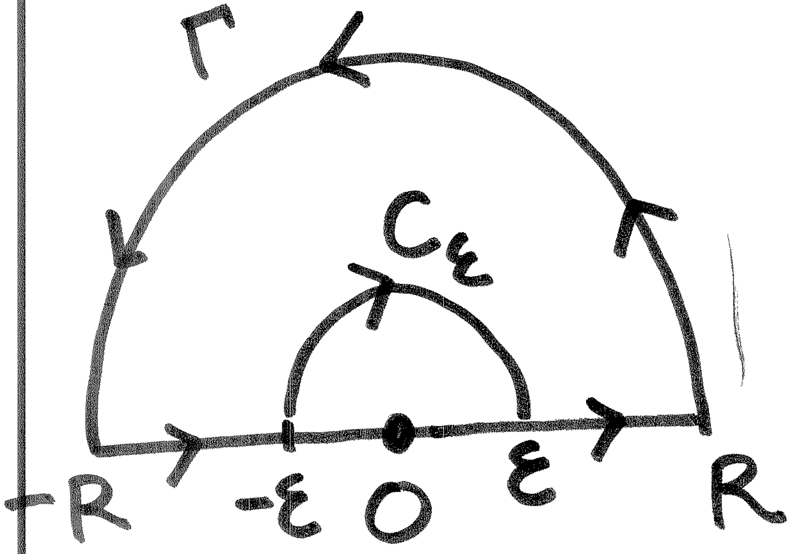
C_ε is the semi circle going clockwise

Ex: Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

Consider $\displaystyle\int_C \frac{e^{iz}}{z}\,dz$

$\dfrac{e^{iz}}{z}$ has a singularity at $z=0$ (on the real line)

simple pole



$$C = \Gamma \cup [-R, -\varepsilon] \cup [\varepsilon, R] \cup C_\varepsilon$$

$$\int_C \frac{e^{iz}}{z} = 0 \quad \text{(Cauchy's Theorem)}$$

$$0 = \underbrace{\cancel{\int_\Gamma}}_{0} + \int_{-R}^{-\varepsilon} + \int_{C_\varepsilon} + \int_{\varepsilon}^{R}$$

$$\frac{e^{ix}-e^{-ix}}{2i} = \sin x$$

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} \frac{e^{iz}}{z}\,dz = -\pi i \operatorname*{Res}_{z=0} \frac{e^{iz}}{z} = -i\pi$$

$$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx = \int_{\varepsilon}^{R} \frac{e^{-ix}}{-x}\,dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx$$

change x to -x

$$= \int_{\varepsilon}^{R} \frac{e^{ix}-e^{-ix}}{x}\,dx = 2i\int_{\varepsilon}^{R} \frac{\sin x}{x}\,dx$$

Check: $\displaystyle\int_{\Gamma} \frac{e^{iz}}{z} \longrightarrow 0, \quad R \longrightarrow \infty \qquad z = Re^{i\theta},\ 0 \le \theta \le \pi$

~~Taking limit $R \to \infty$, $\varepsilon \to 0$~~

$$0 = 2i \lim_{\substack{R\to\infty \\ \varepsilon \to 0}} \int_{\varepsilon}^{R} \frac{\sin x}{x} - i\pi$$

$$\Rightarrow \frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin x}{x}$$