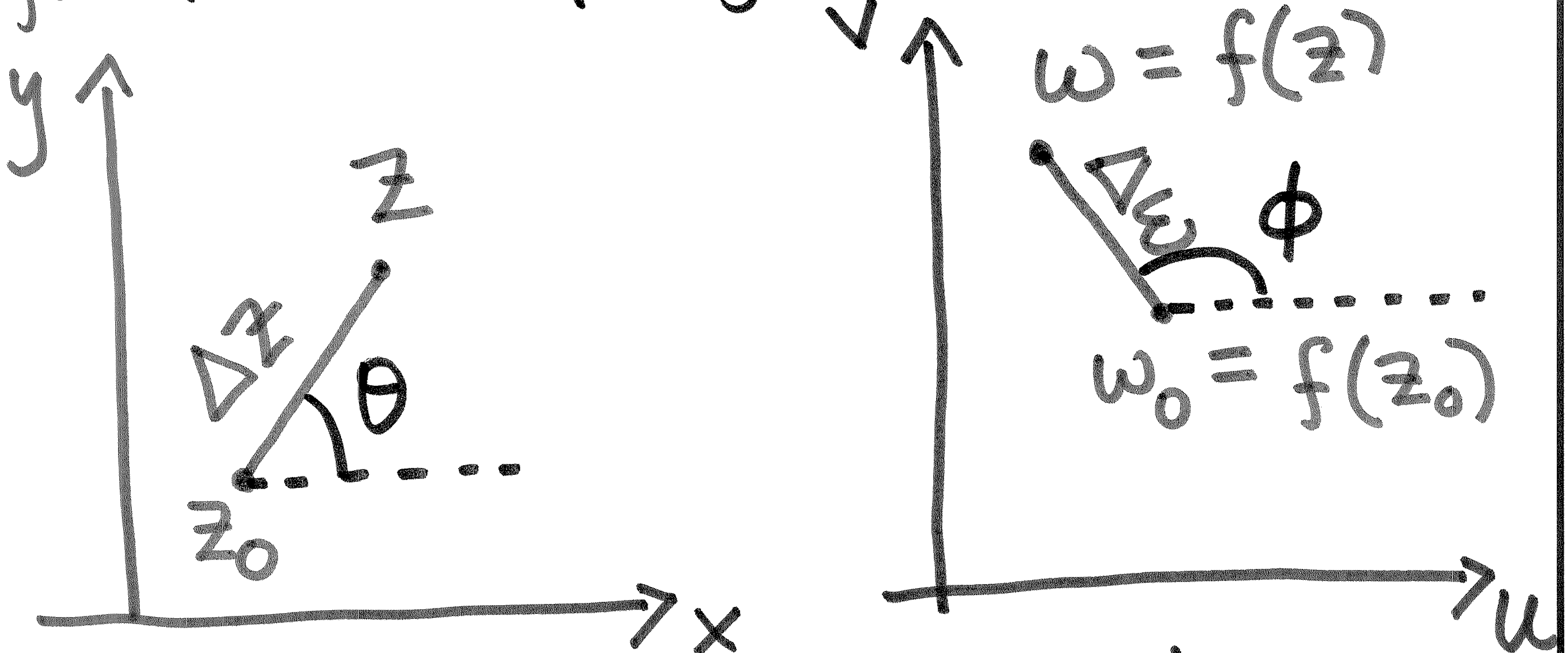
MATH 420

COMPLEX VARIABLES

SESSION no. 33

Conformal Mapping

$z = x + iy$; $w := f(z)$ - analytic function mapping z- to w-plane



z-plane

w-plane
$w = f(z) = u + iv$

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f - analytic; $f'(z_0) = |f'(z_0)| e^{i\alpha}$

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$\Delta z + z_0 = z$; $w_0 + \Delta w = w$

Taylor series about z_0:

$$f(z) = f(z_0) + (z - z_0) f'(z_0)$$

[disregard higher order terms]

$$f(z) - f(z_0) = (z - z_0) f'(z_0)$$

$$\Rightarrow \Delta w = \Delta z \; f'(z_0)$$

$$\Rightarrow |\Delta w| e^{i\phi} = |\Delta z| e^{i\theta} |f'(z_0)| e^{i\alpha}$$

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f: analytic function

$\Rightarrow |\Delta w| = |\Delta z| \, |f'(z_0)|$

$\Rightarrow$ elemental length is magnified by $|f'(z_0)|$ — (a)

& $\phi = \theta + \alpha$

$\Rightarrow$ rotation through α under f ($\alpha = \text{Arg } f'(z_0)$).

— (b)

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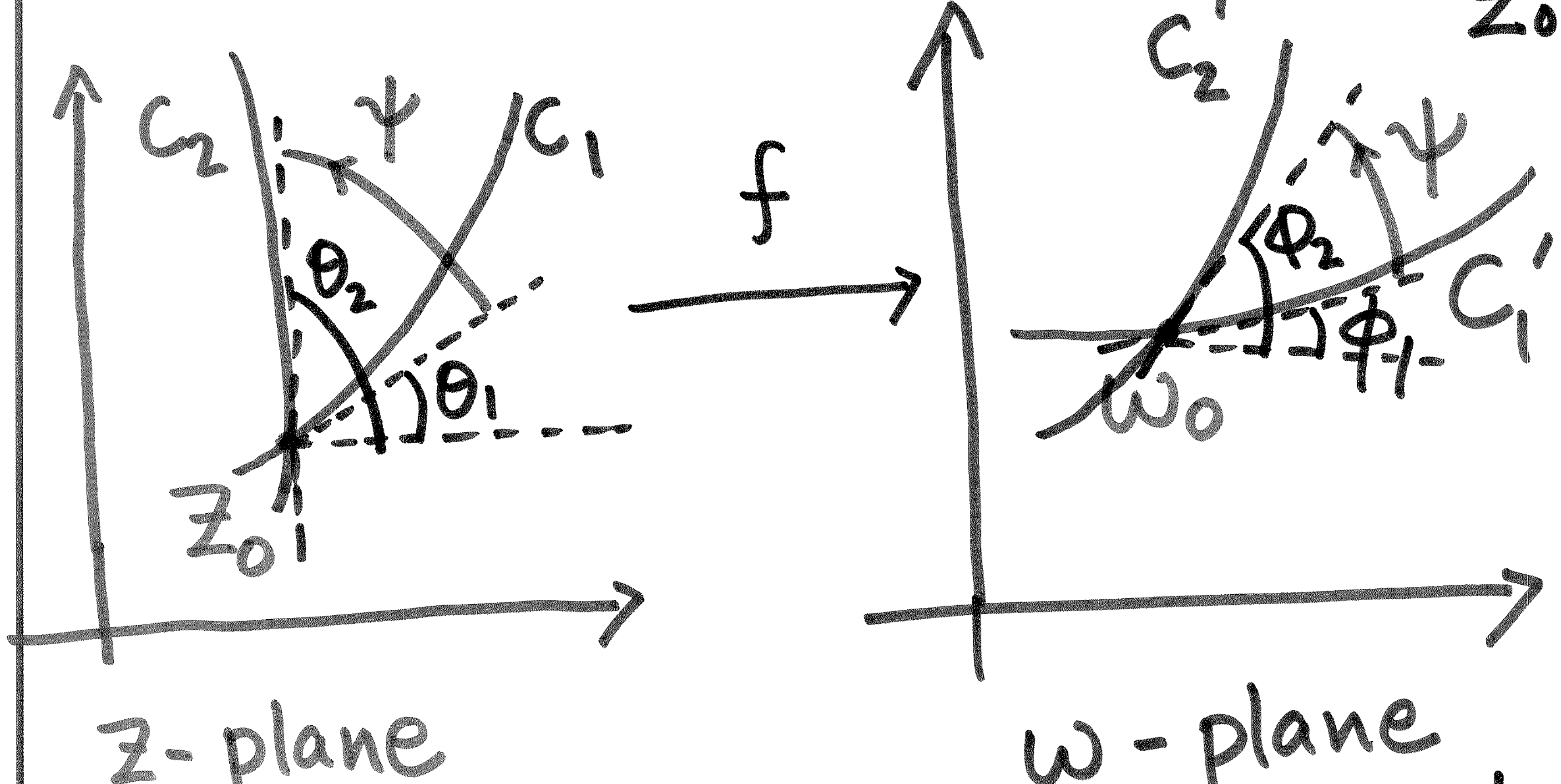
Consider curves C_1 & C_2 through z_0



$$f(C_1) = C_1' \qquad \theta_1 + \alpha = \phi_1$$

$$f(C_2) = C_2' \qquad \theta_2 + \alpha = \phi_2$$

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$\psi = \theta_2 - \theta_1$

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$$\phi_2 - \phi_1 = \theta_2 + \alpha - (\theta_1 + \alpha)$$
$$= \theta_2 - \theta_1 = \psi$$

The angle between C_1 & C_2 is preserved under the mapping f.

A mapping under which the angle between two intersecting curves is preserved both in magnitude and in sense is called conformal. If only the magnitude is preserved then it is called isogonal.

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If f is analytic in some domain D then f is ~~[illegible]~~ conformal in D where $f'(z_0) \neq 0$.

Example: $f(z) = e^z$; f is analytic everywhere; $f'(z) \neq 0$ everywhere $\Rightarrow e^z$ is a conformal map.

$$e^z = e^{x+iy} = e^x e^{iy}$$

$$|e^z| = e^x \; ; \; \operatorname{Arg}(e^z) = y$$

$e^z = e^x e^{iy}$ $\quad e^{c_1} e^{iy}$

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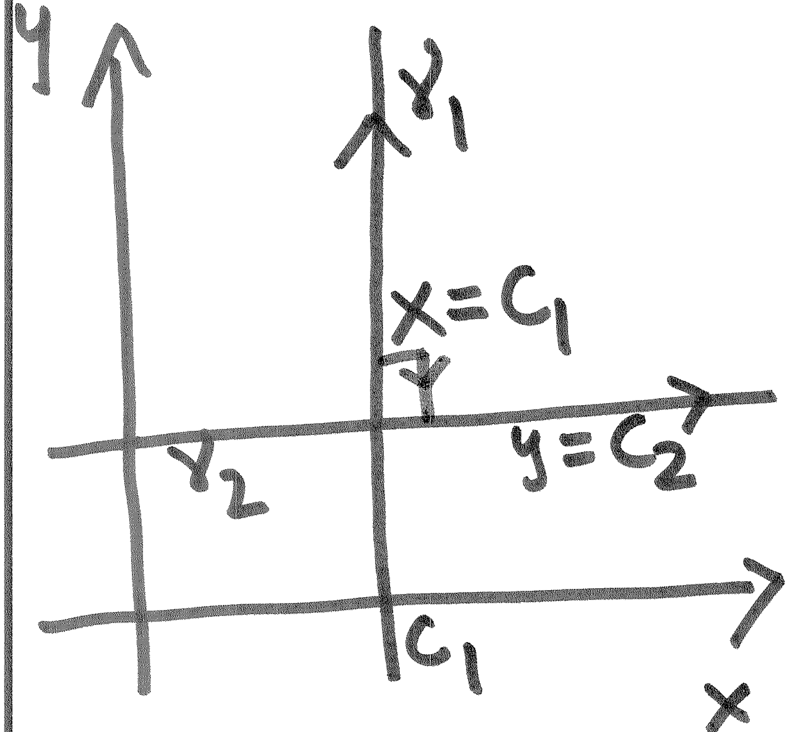

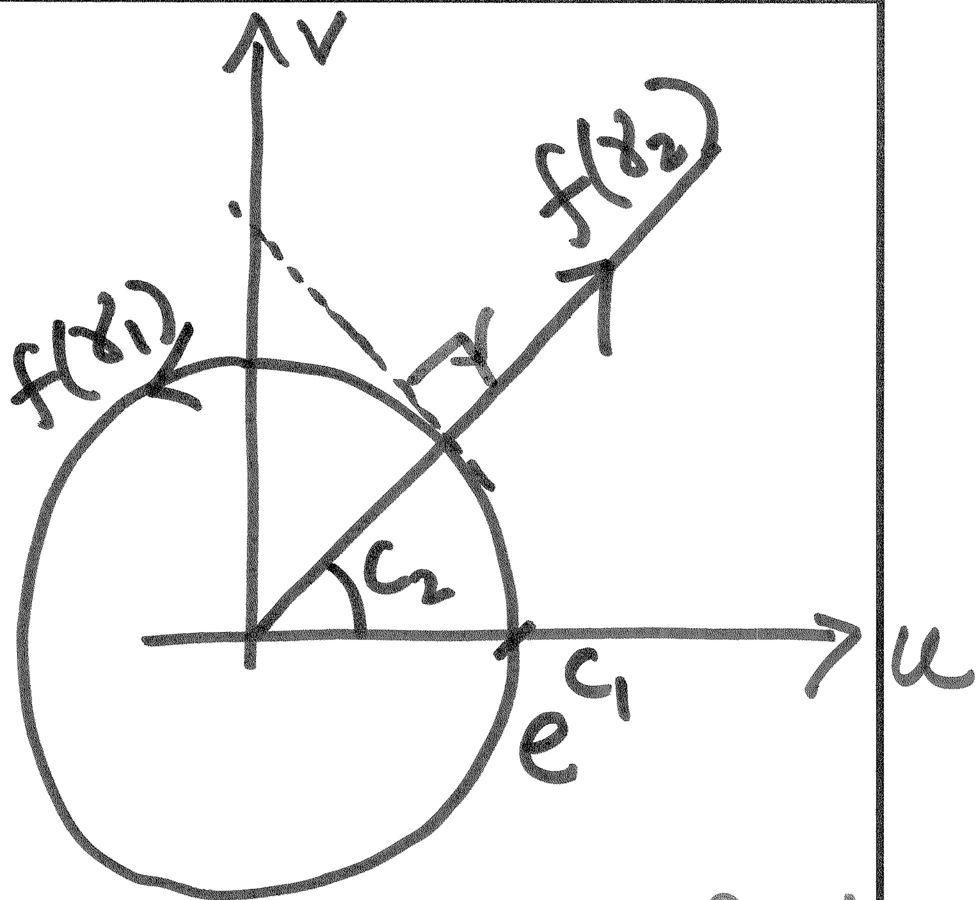

$\gamma_1 : x = C_1$

$\gamma_2 : y = C_2$

Angle between γ_1 & $\gamma_2 =$ -90°

Angle between $f(\gamma_1)$ & $f(\gamma_2) = -90^\circ$