MATH 420

COMPLEX VARIABLES

SESSION no. 35

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Conformal Mapping & Harmonic Functions

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Laplace Eq:

$$H_{xx} + H_{yy} = 0$$

H is harmonic if it solves Laplace Eq.

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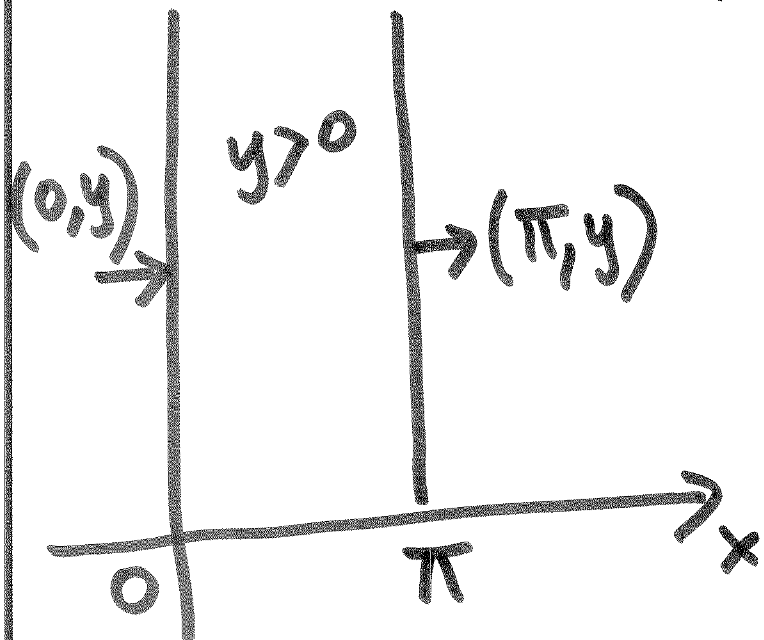
If $f = u + iv$ is analytic then u & v are harmonic

[derived last class]

v: harmonic conjugate of u

Ex: $T(x,y) = e^{-y}\sin x$

$\downarrow$ temp. on a strip



$0 < x < \pi$

$y > 0$

$T_{xx} + T_{yy} = 0$

[shown last class]

$T(0,y) = 0 = T(\pi,y)$

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Ex: $\underbrace{e^{-y}\sin x}_{T(x,y)}$ is the real part of $\underbrace{-ie^{iz}}_{f(z);\ \text{analytic}}$

$$-ie^{iz} = -ie^{i(x+iy)} = -ie^{ix-y}$$

$$= -ie^{-y}e^{ix} = -ie^{-y}(\cos x + i\sin x)$$

$$= \underbrace{e^{-y}\sin x}_{T(x,y)} - ie^{-y}\cos x$$

$e^{-y} \sin x$ is the imaginary part of e^{iz}.

$\hookrightarrow$ analytic

Problem : Given a harmonic function u, find its harmonic conjugate.

Ex: Construct the harmonic conjugate of $u(x,y) = x^3 - 3xy^2$

[Find $v(x,y)$ s.t. $\underbrace{u + iv}_{f}$ is analytic]

$u_x = 3x^2 - 3y^2$; by C-R eqs.

$u_x = v_y = 3x^2 - 3y^2$ — integrate w.r.t. y

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$$u = x^3 - 3xy^2$$

$$V = 3x^2y - y^3 + g(x)$$

$$V_x = 6xy + g'(x) = -u_y \rightarrow \text{CR eq.}$$

$$= 3x(2y)$$

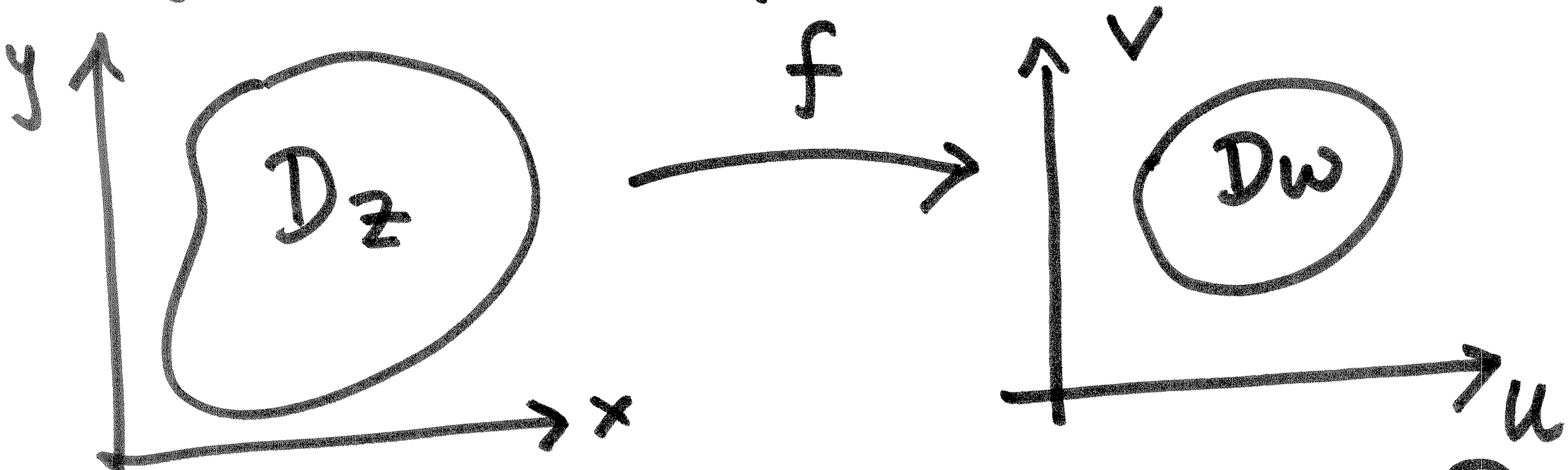
$$= 6xy$$

$$\Rightarrow g'(x) = 0 \Rightarrow g(x) = C$$

$$\Rightarrow V = 3x^2y - y^3 + C$$

↳ harmonic conjugate of v

Thm: $f(z) = u(x,y) + i\,v(x,y)$ is conformal; maps D_z to D_w

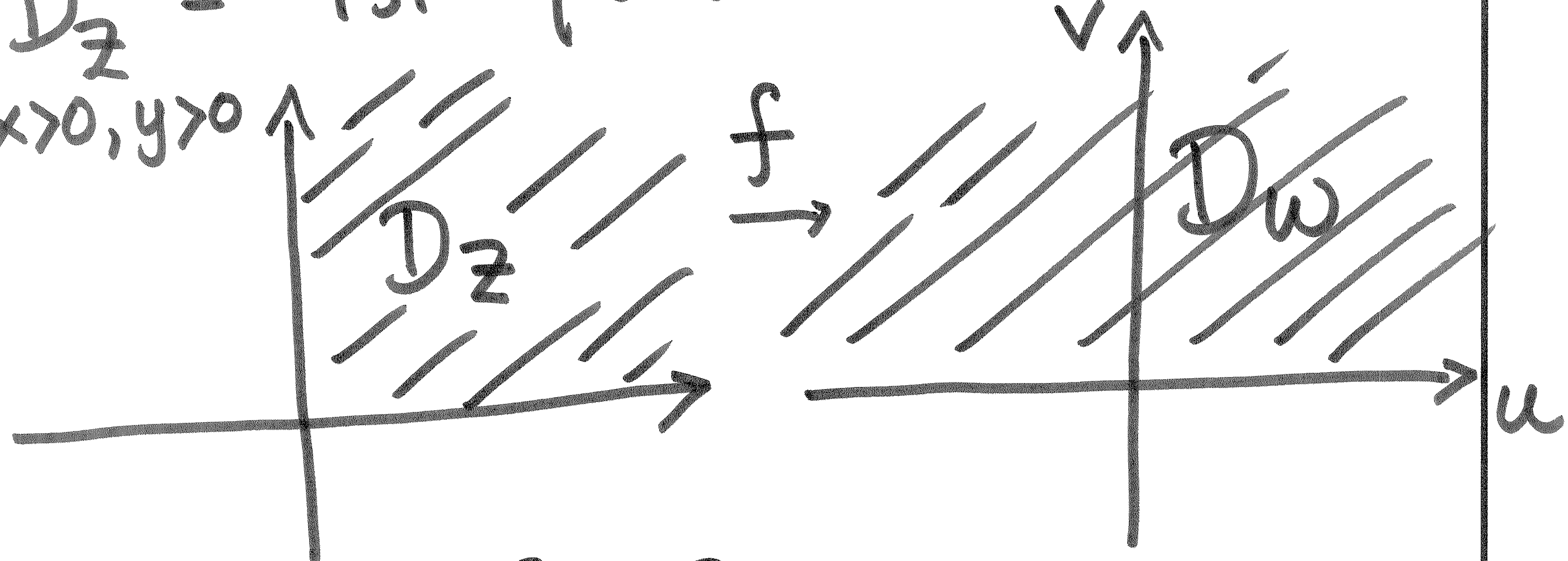


If $h(u,v)$ is harmonic in D_w then $H(x,y) = h(u(x,y), v(x,y))$ is harmonic in D_z.

Ex. $f(z) = z^2 = (x+iy)^2 = \underbrace{x^2-y^2}_{u(x,y)} + i\,\underbrace{2xy}_{v(x,y)}$

D_z = 1st quadrant

$x>0,\ y>0$



$u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$

$h(u,v) = e^{-v} \sin u$ is harmonic in the u-v plane

$h(x^2-y^2, 2xy) = e^{-2xy} \sin(x^2-y^2)$

$= H(x,y)$

H is harmonic in x-y plane.

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Boundary Value Problems

BVP

$$T_{xx} + T_{yy} = 0 \rightarrow \text{equation}$$

and

$$\left.\begin{aligned} T(0,y) &= 0 \\ T(\pi,y) &= 0 \end{aligned}\right\} \text{Boundary conds.}$$

Solution : $e^{-y}\sin x \leftarrow T(x,y)$

Above: Boundary value problem

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Boundary Value Problems

Problem: Solve some

a) partial differential eqn.

such that

b) some boundary conditions are satisfied

of some domain where the solution must be