MATH 420

COMPLEX VARIABLES

SESSION no. 36

Boundary Value Problems:

Dirichlet Problem for Laplace Equation

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Dirichlet Problem

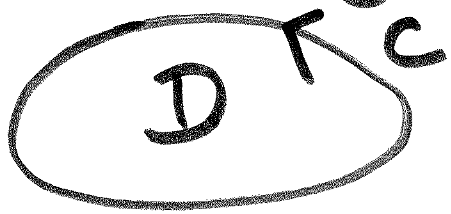
Find u such that

$$u_{xx} + u_{yy} = 0 \quad \text{[Laplace Eq.]}$$

& $u = f$ on C → Bound. Cond.

f → given

where C = boundary of the domain

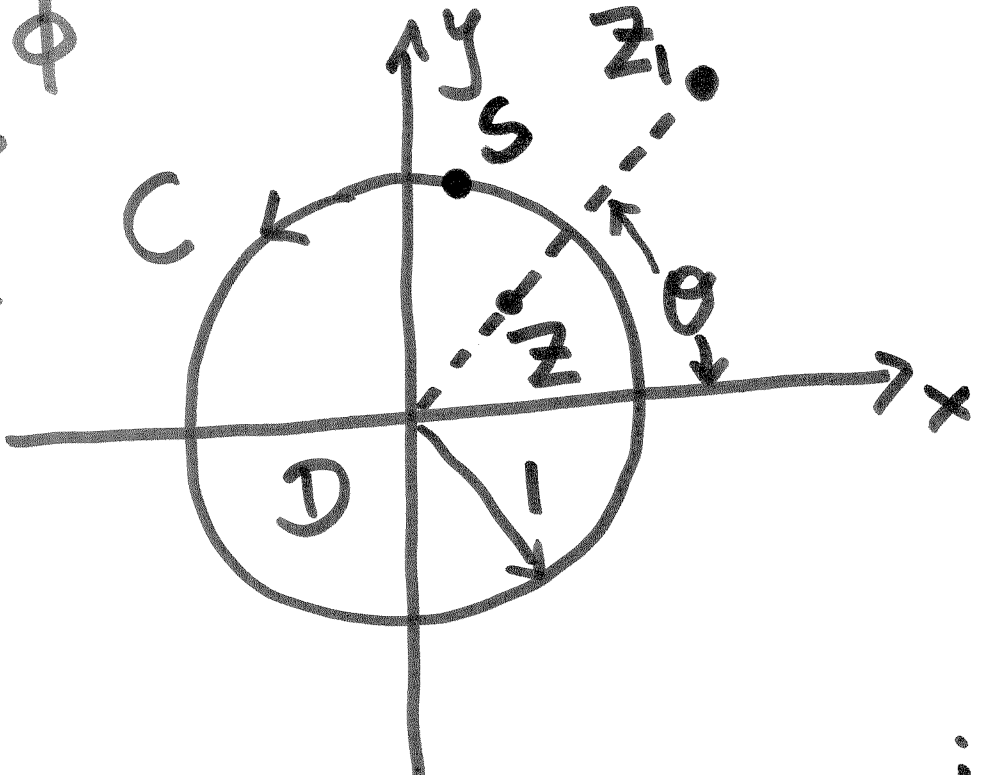


Dirichlet Prob. on the unit circle

C - unit circle, D - interior of C

$s = e^{i\phi}$

$s\bar{s} = 1$



$$z_1 = \frac{1}{r} e^{i\theta}$$

$\frac{1}{r} > 1 \Rightarrow z_1$ is outside C

$$\underset{(x,y)}{z = x + iy} = \underset{(r,\theta)}{r e^{i\theta}}, \quad r < 1$$

Find u that satisfies the Laplace equation

&

for some given h (continuous)

$$u(1,\theta) = h(\theta)$$

boundary condition

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Inside the unit circle

A function that satisfies Laplace eq at (r,θ) in D is given by

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)\,h(\phi)}{1-2r\cos(\theta-\phi)+r^2}\,d\phi\,; \quad r<1$$

$$u(1,\theta) = h(\theta).$$

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Let $u(x,y)$ be harmonic on & inside C. Let $v(x,y)$ is the harmonic conjugate of u.

Then

$f(z) = u(x,y) + i\,v(x,y)$ is analytic on and inside C.

By Cauchy's Integral Formula:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z}\, ds$$

$$\frac{1}{2\pi i} \int_C \frac{f(s)}{s-z_1}\, ds = 0 \quad \text{[Cauchy's Thm]}$$

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z}\, ds - \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z_1}\, ds$$

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$$f(z) = \frac{1}{2\pi i} \int_C \left[\frac{1}{s-z} - \frac{1}{s-z_1} \right] f(s)\, ds$$

$$z_1 = \frac{e^{i\theta}}{r} = \frac{1}{re^{-i\theta}} = \frac{1}{\bar{z}} = \frac{s\bar{s}}{\bar{z}}$$

$$\frac{1}{s-z} - \frac{1}{s-z_1} = \frac{1}{s-z} - \frac{1}{s - \frac{s\bar{s}}{\bar{z}}}$$

$$= \frac{s\bar{s} - z\bar{z}}{s|s-z|^2}$$

verify (HW)

$$f(z) = \frac{1}{2\pi i} \int_C \frac{s\bar{s} - z\bar{z}}{s|s-z|^2} f(s)\, ds \qquad s = e^{i\varphi} \Rightarrow ds = ie^{i\varphi} d\varphi$$

Convert to polar

$$f(re^{i\theta}) = \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{1-r^2}{\cancel{e^{i\varphi}}(1+r^2-2r\cos(\theta-\varphi))} \frac{f(e^{i\phi})\, \cancel{i}\cancel{e^{i\phi}}}{} d\varphi$$

$$|s-z|^2 \underset{\text{HW}}{=} 1+r^2-2r\cos(\theta-\varphi)$$

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)\, f(e^{i\phi})}{1+r^2-2r\cos(\theta-\phi)}\, d\varphi$$

Equating the real part

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2 \quad u(e^{i\varphi})}{1+r^2-2r\cos(\theta-\varphi)}\, d\varphi$$

Given $u(e^{i\varphi}) = h(\varphi)$