MATH 420

COMPLEX VARIABLES

SESSION no. 37

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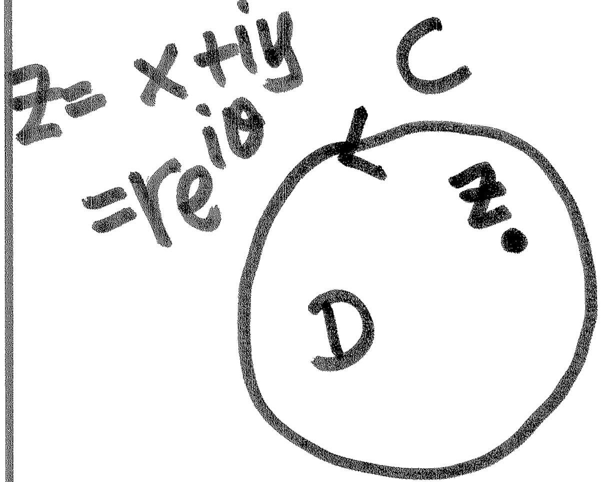
Boundary Value Problems

- Dirichlet Problem

Dirichlet Problem inside an unit circle



C: unit circle

Find $u(x,y)$ such that for a given f:

$$u_{xx} + u_{yy} = 0$$

and $u = f$ on C

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The Solution:

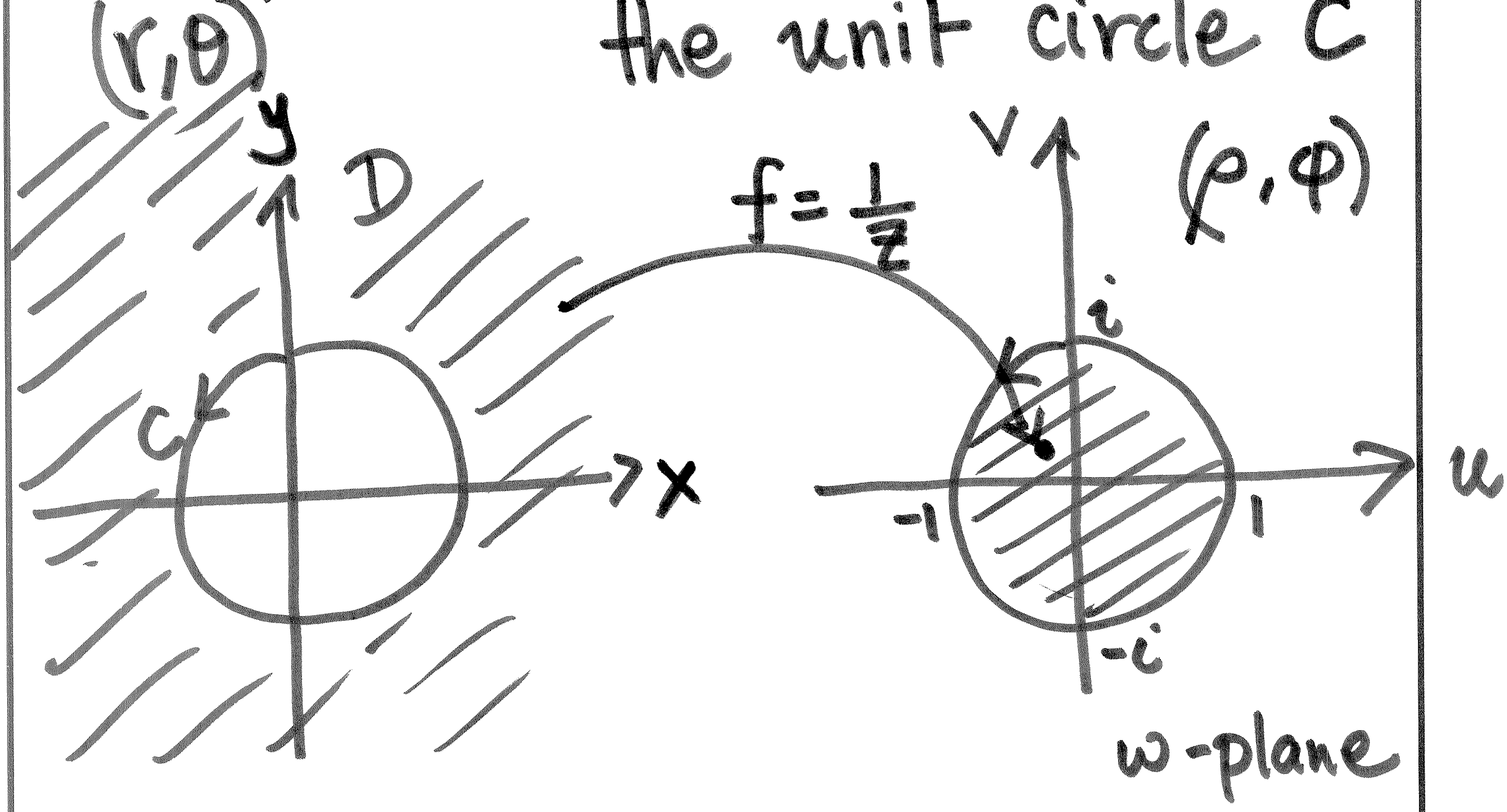
$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)} f(\phi)\,d\phi$$

$r < 1$

$$u(1,\theta) = f(\theta)$$

$r=1.$ [Proved last class]

Dirichlet Problem for outside the unit circle C



$$w = f(z) = \frac{1}{z}$$

$f(z) = \frac{1}{z}$ is analytic and hence conformal for $z \neq 0$.

$z = 0$ in not interesting.

f is conformal everywhere in D $(0 \notin D)$.

$z = re^{i\theta}$. $w = f(z) = \rho e^{i\phi}$

Also $w = \frac{1}{z} = \frac{1}{r} e^{-i\theta}$

$r > 1 \Rightarrow \frac{1}{r} < 1 \Rightarrow w$ is inside the unit circle.

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$$\frac{1}{r} e^{-i\theta} = \rho e^{i\phi}$$

$$\rho = \frac{1}{r}, \quad \phi = -\theta$$

The solution inside C : $\rho < 1$

$$u(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-\rho^2)\, u(1,\psi)}{1+\rho^2-2\rho\cos(\phi-\psi)}\, d\psi$$

Change to (r,θ) by using $\rho = \dfrac{1}{r}$, $\phi = -\theta$.

$\tilde{u}(r,\theta)$

$$u\left(\frac{1}{r}, -\theta\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(1-\frac{1}{r^2}\right)u(1,\psi)}{1+\frac{1}{r^2}-\frac{2}{r}\cos(\theta+\psi)}\,d\psi$$

$$= -\frac{1}{2\pi}\int_0^{2\pi}\frac{(r^2-1)\,u(1,\psi)}{r^2+1-2r\cos(\theta+\psi)}\,d\psi$$

$\hookrightarrow$ solution of the Dirichlet problem outside C.

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$\tilde{u}(r,\theta) = u\left(\frac{1}{r}, -\theta\right)$

is harmonic in D

because under a conformal map like $f(z) = \frac{1}{z}$,

harmonic functions remain harmonic.

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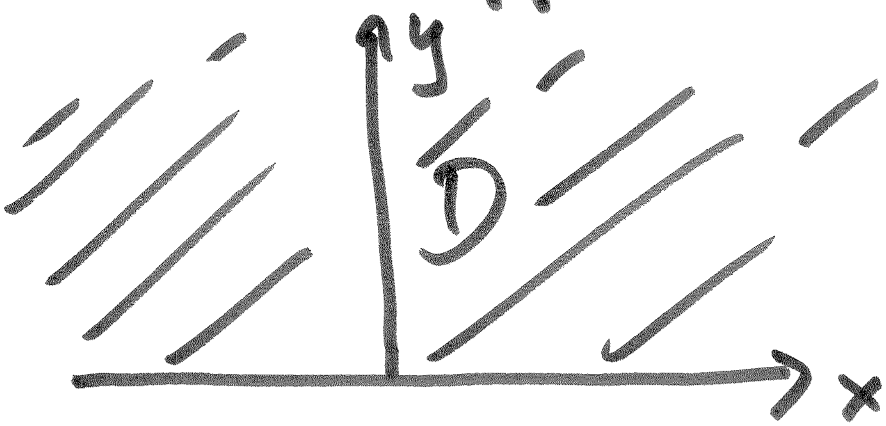
Next: Solve the Dirichlet problem in the upper half plane:



Find $u(x,y)$ s.t. for a given f

$u_{xx} + u_{yy} = 0$ in D &

boundary

$u = f$ on the x-axis