MATH 420

COMPLEX VARIABLES

SESSION no. 38

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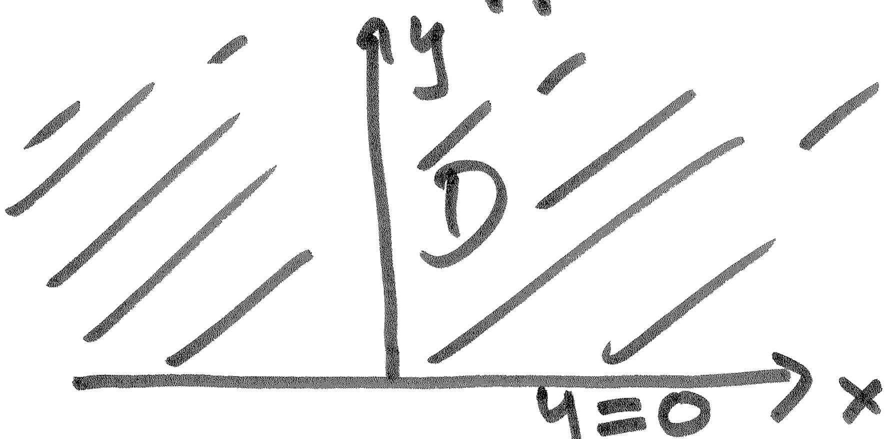
Solve the Dirichlet problem in the upper half plane:



Find $u(x,y)$ s.t. for a given f

$u_{xx} + u_{yy} = 0$ in D &

$u = f$ on the x-axis

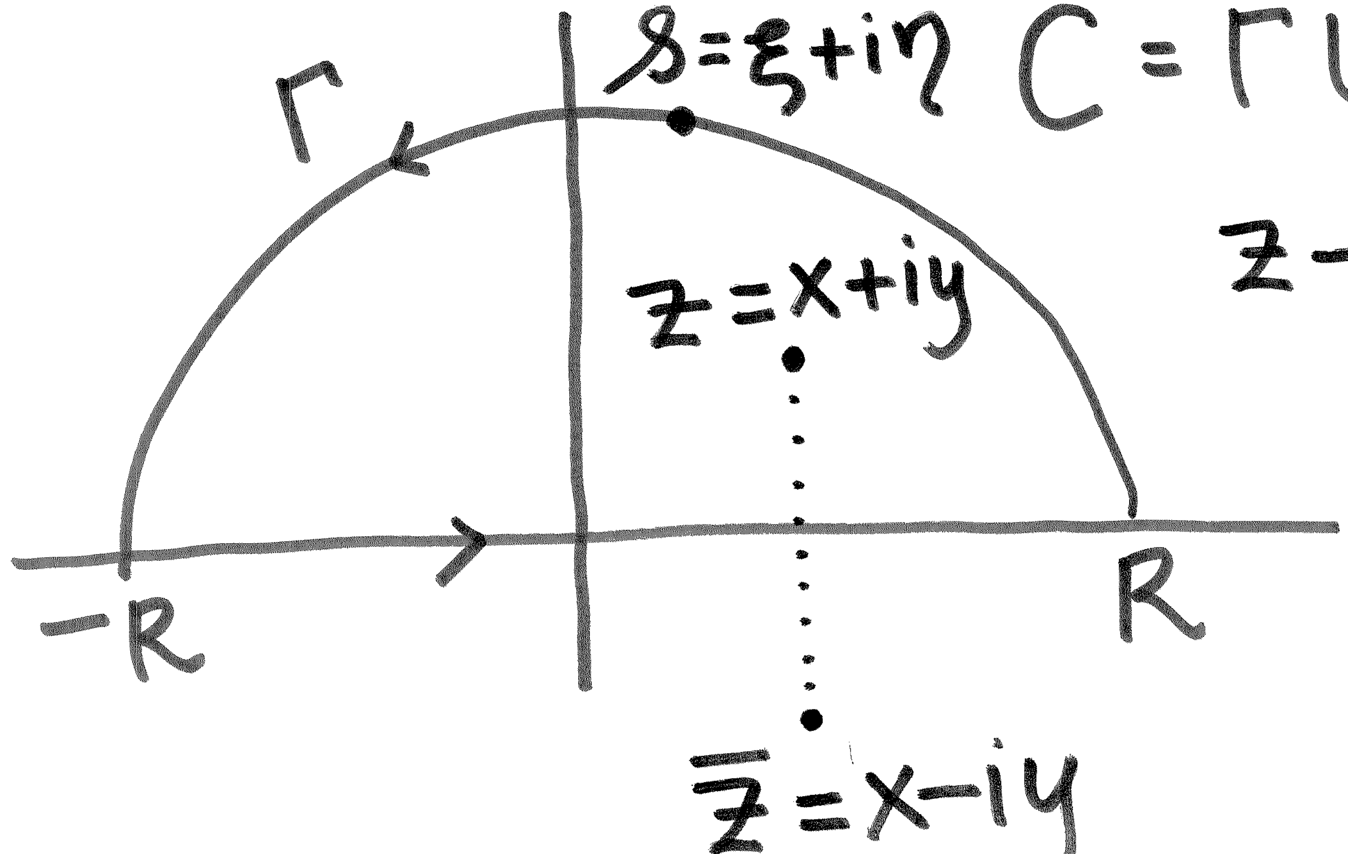
boundary $y=0$

$u(x,0)$ is given

Solution. Let $f = u + iv$ be analytic. Find $u(x,y)$.

$C = \Gamma \cup [-R, R]$

$z - \bar{z} = 2iy$



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$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}\,ds$$ Cauchy's Formula

$$\frac{1}{2\pi i}\int_C \frac{f(s)}{s-\bar{z}}\,ds = 0$$ Cauchy's Thm

Subtracting ~~Adding~~,

$$f(z) = \frac{1}{2\pi i}\int_C \left[\frac{1}{s-z} - \frac{1}{s-\bar{z}}\right] f(s)\,ds$$

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$$f(z) = \frac{1}{2\pi i} \int_C \frac{z - \bar{z}}{(s-z)(s-\bar{z})} f(s)\, ds$$

$s = \xi$

$$= \frac{1}{\cancel{2}\pi \cancel{i}} \left[\int_{-R}^{R} \frac{\cancel{2}\cancel{i} y\, f(\xi, 0)}{(\xi - x)^2 + y^2} d\xi + \int_\Gamma \frac{\cancel{2}\cancel{i} y\, f(s)\, ds}{(s-z)(s-\bar{z})} \right]$$

$R \to \infty$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y f(\xi, 0)}{(\xi - x)^2 + y^2} d\xi + \frac{1}{\pi} \underbrace{\int_\Gamma \frac{y f(s)\, ds}{(s-z)(s-\bar{z})}}_{\to 0}$$

assuming f is bounded

Real ~~parts~~ part on both sides :

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\,u(\xi,0)}{(\xi-x)^2+y^2}\,d\xi$$

$\downarrow$

Solution to the Dirichlet problem in the upper half plane

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Poisson's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad : \text{Laplace Eq.}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = h(x,y) \quad :$$

Poisson's Equation

$$z = x + iy \;;\quad \bar{z} = x - iy$$

$$x = \frac{z + \bar{z}}{2} \;;\quad y = \frac{z - \bar{z}}{2i}$$

chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial u}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} + \frac{\partial u}{\partial \bar{z}}$$

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$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial u}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial y}$$

$$= i\left[\frac{\partial u}{\partial z} - \frac{\partial u}{\partial \bar{z}}\right]$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2}$$

$$\frac{\partial^2}{\partial y^2} = -\left(\frac{\partial^2}{\partial z^2} - 2\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2}\right)$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

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Solve: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 8(x^2 - y^2)$

After changing to z, $\bar{z}$

$$\cancel{4}\frac{\partial^2 u}{\partial z \partial \bar{z}} = 8\left(\left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2\right)$$

$$= 2\left((z+\bar{z})^2 - (z-\bar{z})^2\right)$$

$$\cancel{4}\frac{\partial^2 u}{\partial z \partial \bar{z}} = \cancel{4}(z^2 + \bar{z}^2)$$

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = h(x, y) \text{ becomes}$$

$$4 \frac{\partial^2 u}{\partial z \partial \bar{z}} = h\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

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Integrating w.r.t. z & $\bar{z}$ successively,

$$u(z,\bar{z}) = \frac{z^3\bar{z}}{3} + \frac{\bar{z}z^3}{3} + F(z) + G(\bar{z})$$

Back to (x,y) → particular soln. → solution to Laplace eq.

$$u(x,y) = \frac{2}{3}(x^4 - y^4) + F(x+iy) + G(x-iy)$$