MATH 420

COMPLEX VARIABLES

SESSION no. 39

University of Idaho

Homework 7

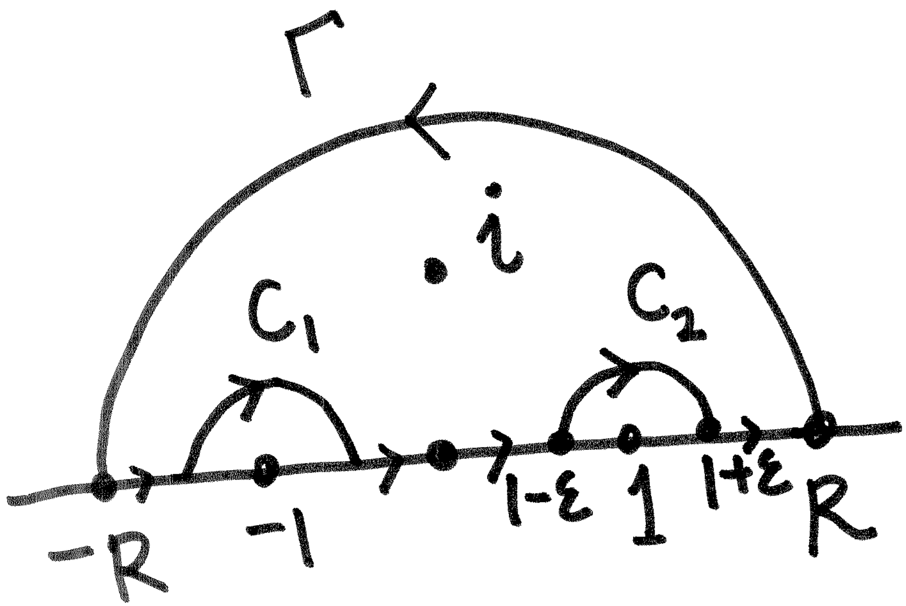
1. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^4-1}\,dx$

$$f(z) = \frac{z}{z^4-1}$$

singularities :

$\pm 1, \pm i$

simple poles



$f(z) = z / z^4 - 1$

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$$\int_C f = \int_\Gamma + \int_{-R}^{-1-\varepsilon} + \int_{C_1} + \int_{-1+\varepsilon}^{1-\varepsilon} + \int_{C_2} + \int_{1+\varepsilon}^{R}$$

$$\int_C f = 2\pi i \operatorname*{Res}_{z=i} f = -\frac{\pi i}{2}$$ (by Residue Thm)

$$\operatorname*{Res}_{z=i} f = \left.\frac{z}{4z^3}\right|_{z=i} = -\frac{1}{4}$$

$$\operatorname*{Res}_{z=1} f = 1/4, \quad \operatorname*{Res}_{z=-1} f = 1/4$$

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$$\int_{C_1} = -\pi i \operatorname*{Res}_{z=-1} = -\frac{\pi i}{4}$$

$$\int_{C_2} = -\pi i \operatorname*{Res}_{z=1} = -\pi i/4$$

$$R \to \infty, \quad \varepsilon \to 0, \quad \int_\Gamma \to 0$$

$$-\frac{\pi i}{2} = \int_{-\infty}^{\infty} \frac{x}{x^4-1}\,dx - \frac{\pi i}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x}{x^4-1}\,dx = 0.$$

2.

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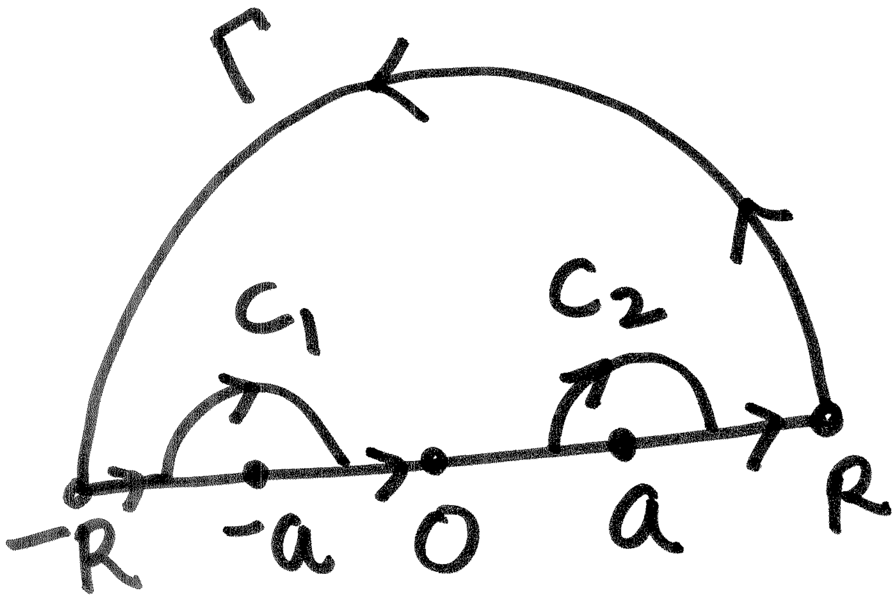
$$\int_0^\infty \frac{\cos x}{a^2 - x^2}$$

$$f(z) = \frac{e^{iz}}{a^2 - z^2}$$

singularities

$z = \pm a$



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$$\oint_C f = \int_\Gamma + \int_{-R}^{-a-\varepsilon} + \int_{C_1} + \int_{-a+\varepsilon}^{a-\varepsilon} + \int_{C_2} + \int_{a-\varepsilon}^{R}$$

$$\oint_C f = 0 \quad \left[\text{by Cauchy's Theorem}\right]$$

$$\int_{C_1} = -\pi i \operatorname*{Res}_{z=-a} = -\frac{e^{ia}}{2a}(-\pi i)$$

$$\int_{C_2} = -\pi i \operatorname*{Res}_{z=a} = \frac{e^{-ia}}{2a}(-\pi i)$$

$R \to \infty,\ \varepsilon \to 0,\ \int_\Gamma \to 0$

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$$0 = \int_{-\infty}^{\infty} \frac{e^{ix}}{a^2 - x^2} + \underbrace{\frac{\pi}{2a} i \left(e^{-ia} + e^{ia} \right)}_{-\frac{\pi}{a} \sin a}$$

$$+\frac{\pi}{a} \sin a = \int_{-\infty}^{\infty} \frac{e^{ix}}{a^2 - x^2} dx$$

Equate the real part on both sides

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$$\frac{\pi}{a}\sin a = \int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\,dx$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2} = 2\int_{0}^{\infty} \frac{\cos x}{a^2 - x^2} = \frac{\pi}{a}\sin a$$

$$\Longrightarrow \int_{0}^{\infty} \frac{\cos x}{a^2 - x^2} = \frac{\pi}{2a}\sin a$$

not conformal at $1, -2$

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3. $f(z) = 2z^3 + 3z^2 - 12z$

Analytic everywhere.

Critical points: $f'(z) = 0$

$\Rightarrow 6z^2 + 6z - 12 = 0$

$\Rightarrow z^2 + z - 2 = 0$

$\Rightarrow (z-1)(z+2) = 0$

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4

$$H(x,y) = e^{x}\cos y \rightarrow u(x,y)$$

Check $H_{xx} + H_{yy} = 0$

Find $v(x,y)$ such that
$u + iv$ is analytic.

Use C-R eqs; integrate ...

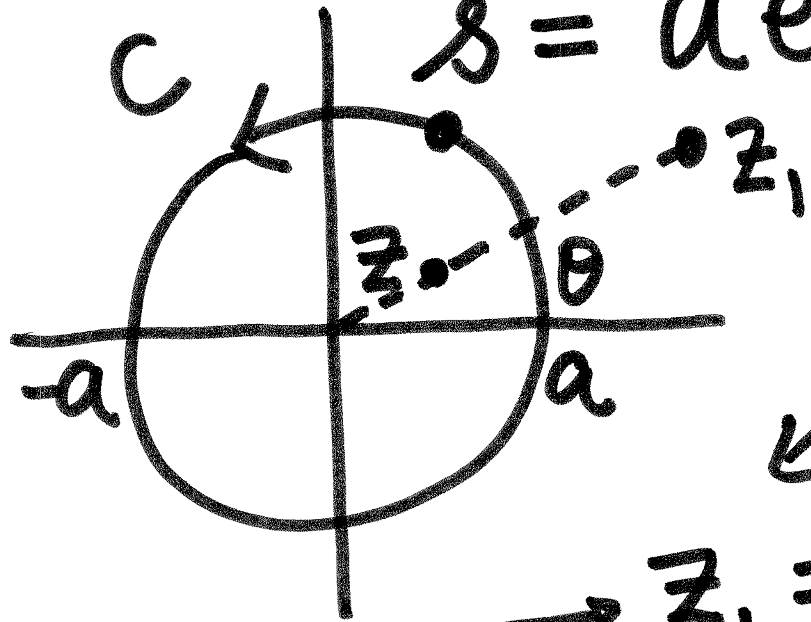
$$v(x,y) = e^{x}\sin y + C$$

Dirichlet problem for inside a circle : $|z| = a \leftarrow$ arbitrary.

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2-r^2)\, h(\varphi)\, d\varphi}{a^2 - 2ar\cos(\theta-\varphi) + r^2}, \quad r < a$$

$$u(a,\theta) = h(\theta) \quad ; \; r = a$$

$s = ae^{i\phi}$



$z = re^{i\theta}$;

$r < a$

$\rightarrow z_1 = \frac{a^2}{r} e^{i\theta}$

$|z_1| > a$ since $r < a$