MATH 420

COMPLEX VARIABLES

SESSION no. 40

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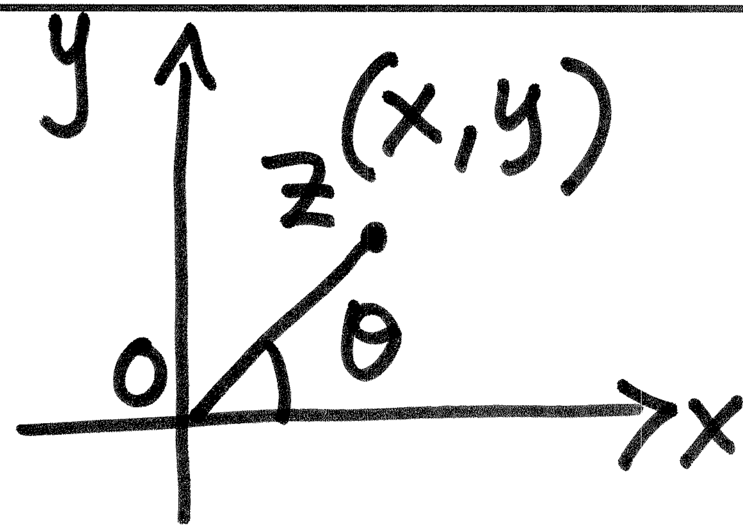
Review

$$z = x + iy$$

$$\bar{z} = x - iy$$



$$|z| = |\bar{z}|$$

$$z\bar{z} = |z|^2 = |\bar{z}|^2$$

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$$z = |z|\, e^{i\theta}$$

$$= |z| (\cos\theta + i \sin\theta)$$

$$i = e^{i(\pi/2 + 2n\pi)}$$

$$-1 = e^{i(\pi + 2n\pi)}$$

$$1 + i = \sqrt{2}\, e^{i\pi/4}$$

$$\sqrt{-1} = i \quad \text{(definition)}$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

Practice Problems

#4 a) $\operatorname{Log}(i^5) \stackrel{\checkmark}{=} \operatorname{Log}(i) \stackrel{\checkmark}{=} \dfrac{i\pi}{2}$

$i^5 = i^4 \cdot i = i$

$\operatorname{Log}(z) = \underset{\wedge}{\overset{\log}{}}|z| + i\operatorname{Arg}(z)$

$$\operatorname{Log}(i) = \log 1 + i\frac{\pi}{2} = i\,\pi/2$$

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4 b) $5\,\mathrm{Log}(i) = 5\left(i\frac{\pi}{2}\right) = i\,\frac{5\pi}{2}$

In reals $\log x^a = a \log x$

but not necessarily for complex nos.

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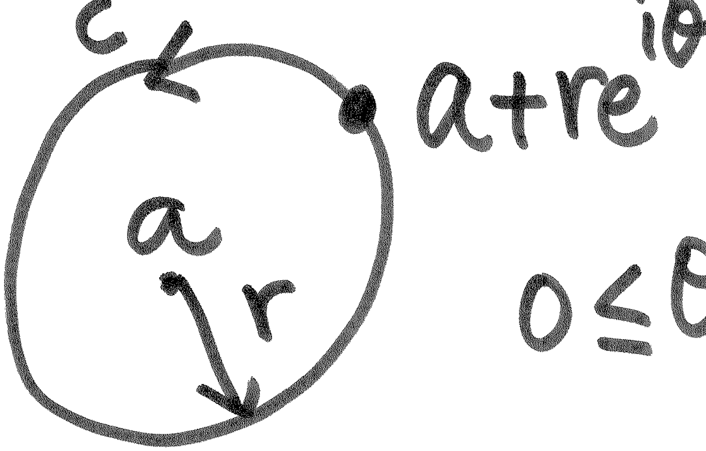
#7. Find $\frac{1}{2\pi}\int_0^{2\pi} \sin^2\left(\frac{\pi}{6}+2e^{i\theta}\right)d\theta$

$$\left[f(a)=\frac{1}{2\pi}\int_{C\,0}^{2\pi} f(a+re^{i\theta})\,d\theta\right] \text{Gauss MVT}$$



$0 \leq \theta \leq 2\pi$

$$\frac{1}{2\pi}\int_0^{2\pi} \sin^2\Big(\overbrace{\frac{\pi}{6}+2e^{i\theta}}^{\text{Circle with center } \pi/6}\Big)\,d\theta$$

Gauss MVT

$$= \sin^2\left(\frac{\pi}{6}\right)$$

$$= \frac{1}{4}$$

8(a) Find the Laurent series of

$$(z-3)\sin\frac{1}{z+2}\;;\quad z=-2.$$

$$\big(z+2-5\big)\sin\frac{1}{z+2}$$

$$=\big(z+2-5\big)\left[\frac{1}{z+2}-\frac{1}{3!\,(z+2)^3}+\frac{1}{5!\,(z+2)^5}-\cdots\right]$$

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$$(z+2)\left[\frac{1}{z+2}-\frac{1}{3!(z+2)^3}+\cdots\right]$$

$$-5\left[\frac{1}{z+2}-\frac{1}{3!(z+2)^3}+\cdots\right]$$

$$=1-\frac{5}{z+2}-\frac{1}{3!(z+2)^2}+\frac{5}{3!(z+2)^3}\cdots$$

$\Rightarrow z=-2$ is an essential singularity.

(*) Can also do this by Integral For. 10 & ~~de~~ formula for derivatives

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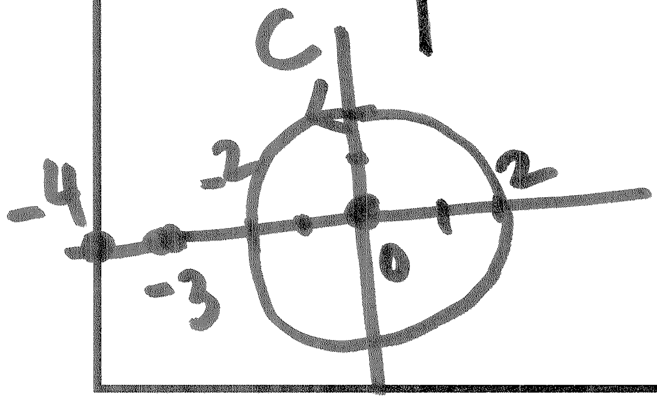
#9 (a) $\int_C \frac{dz}{z^3(z+4)}$ (*)

a) $C: |z| = 2$

Singularities are @ $z = 0, -4$

pole of order 3

Simple pole



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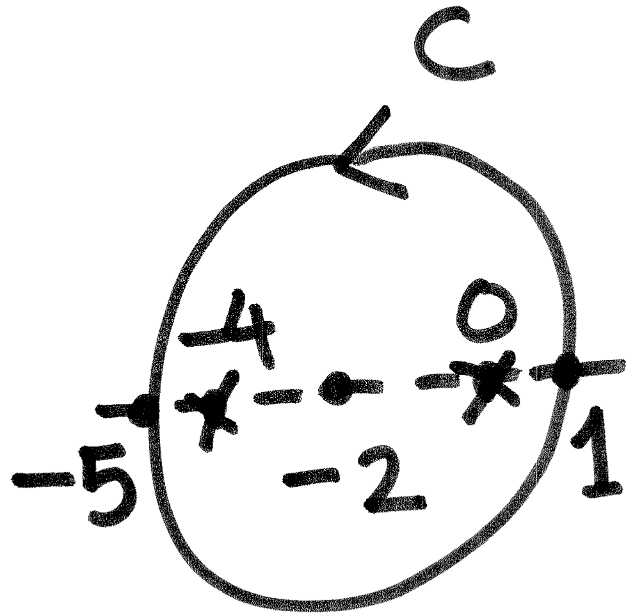
$$\int_C \frac{dz}{z^3(z+4)} = 2\pi i \operatorname*{Res}_{z=0} = \frac{2\pi i}{4^3} = \boxed{\frac{\pi i}{32}}$$

$$\operatorname*{Res}_{z=0} \frac{1}{z^3(z+4)} = \lim_{z\to 0} \frac{1}{(3-1)!} \frac{d^2}{dz^2}\left(\cancel{z^3} \frac{1}{\cancel{z^3}(z+4)}\right)$$

$$= \frac{1}{\cancel{2}} \lim_{z\to 0} \frac{\cancel{2}}{(z+4)^3} = \frac{1}{4^3}$$

9(b) $$\int_C \frac{dz}{z^3(z+4)} = 2\pi i \left(\operatorname*{Res}_{z=0} + \operatorname*{Res}_{z=-4} \right)$$



$C: \; |z+2| = 3$

$$\operatorname*{Res}_{z=-4} = \lim_{z\to -4} (z+4)\frac{1}{z^3(z+4)} = \frac{1}{(-4)^3}$$

$$\int_C \frac{dz}{z^3(z+4)} = 2\pi i\left[\frac{1}{4^3} + \frac{1}{(-4)^3}\right]$$

$$= 0$$