

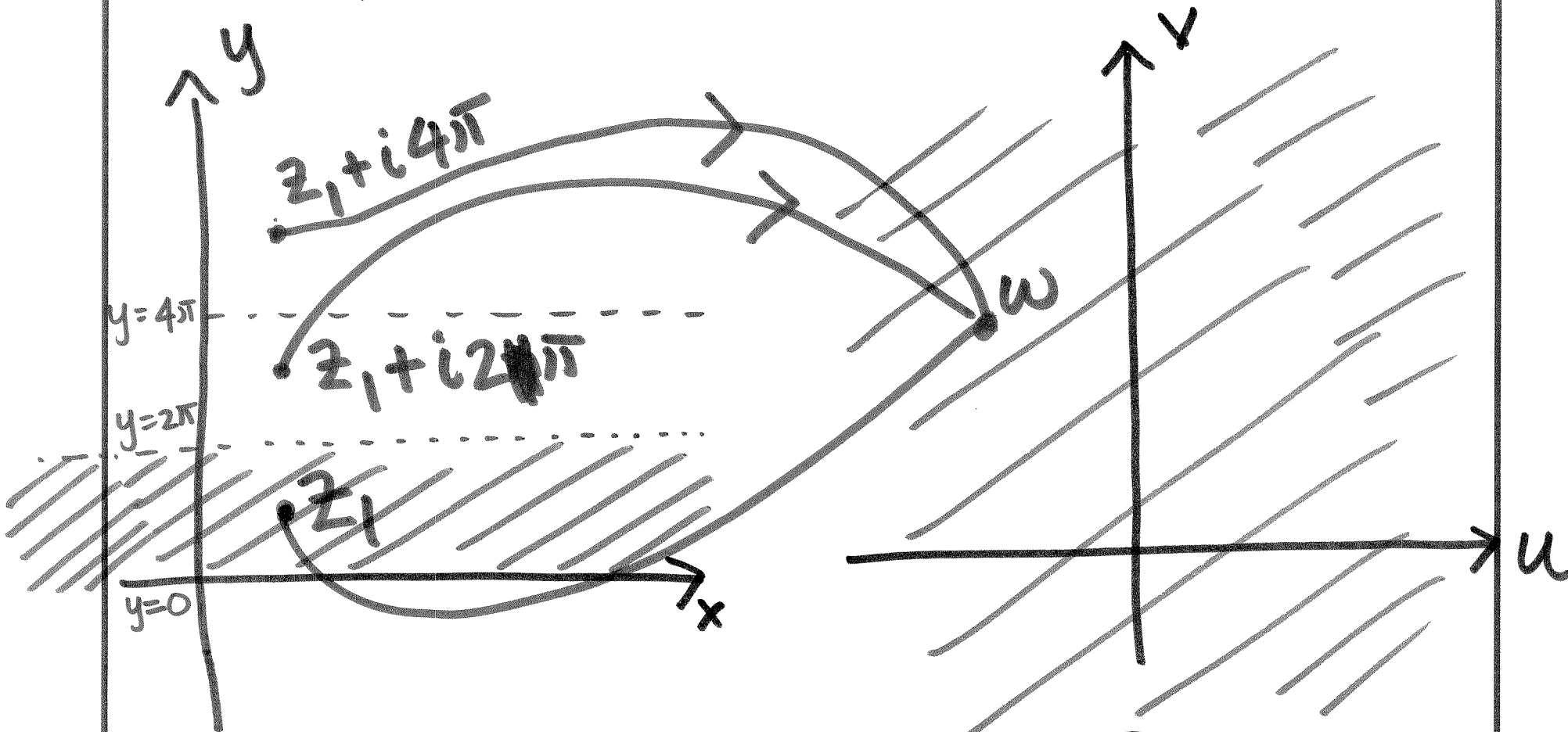
MATH 420

COMPLEX VARIABLES

SESSION no. 12

Exp & Log functions

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z -plane

$e^z = w$ plane

exp : many-one map.

$$z = \text{Log } w = \log |w| + i (\text{Arg } w + 2n\pi)$$

$n=0$ Principal value

$$\text{Log } w = \log |w| + i \text{Arg } w$$

$$0 \leq \text{Arg } w < 2\pi$$

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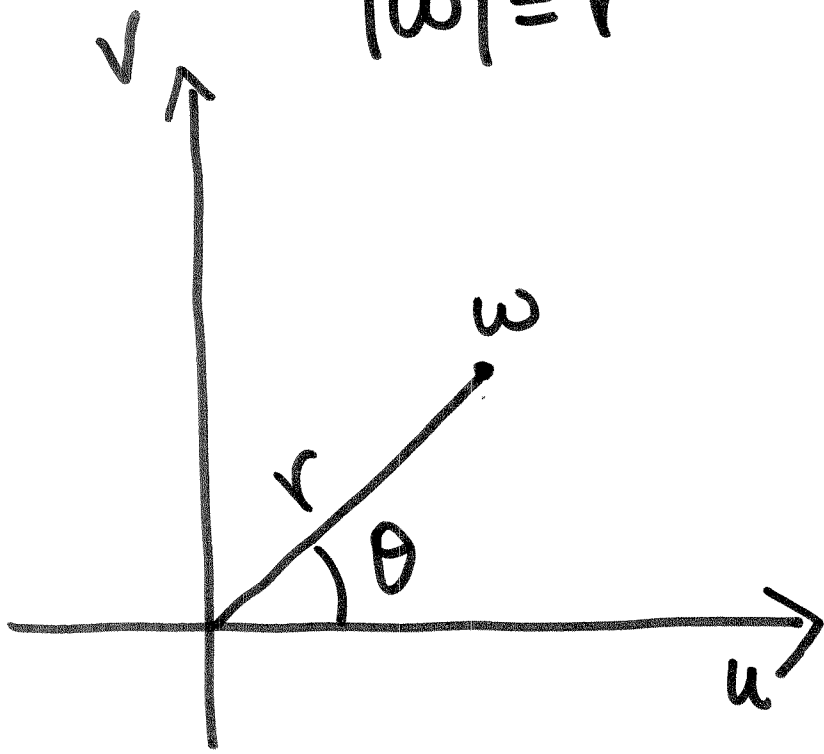
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$$\log w = \log |w| + i (\text{Arg } w + 2n\pi)$$

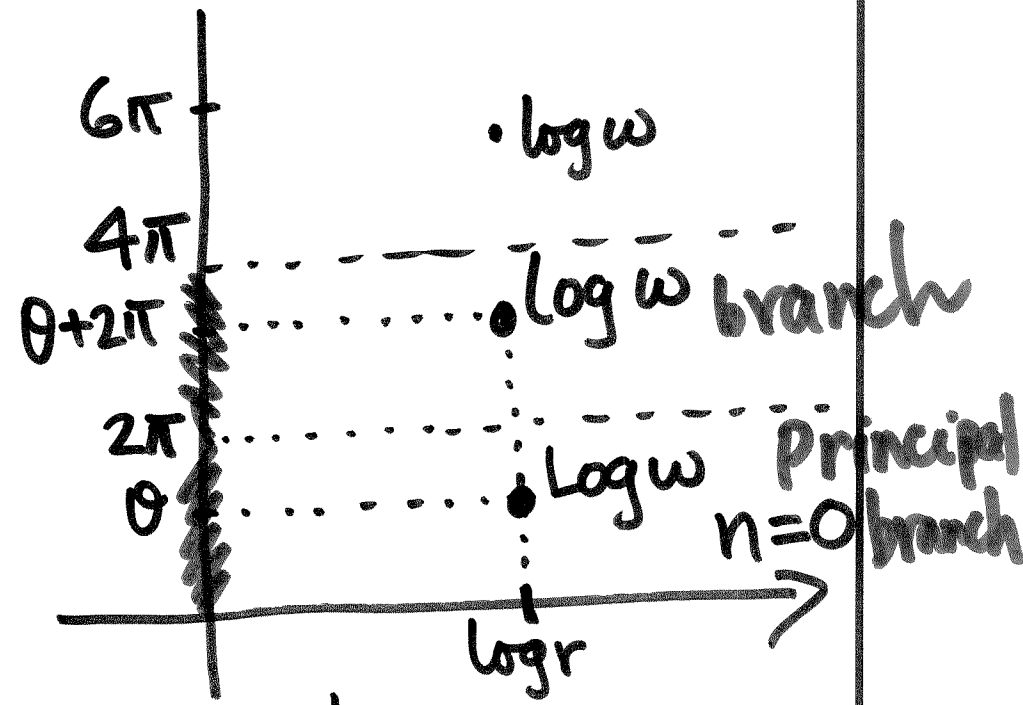
$$w = r e^{i\theta}$$

$$z = \log w$$

$$|w| = r$$



w-plane



$$z = \log w$$

$n=0$: Principal value

$$\text{Example: } \text{Log}(-1 - \sqrt{3}i)$$

$$= \log 2 + i\left(\frac{4\pi}{3} + 2n\pi\right)$$

$$\text{Log} = \log 2 + i \frac{4\pi}{3}$$

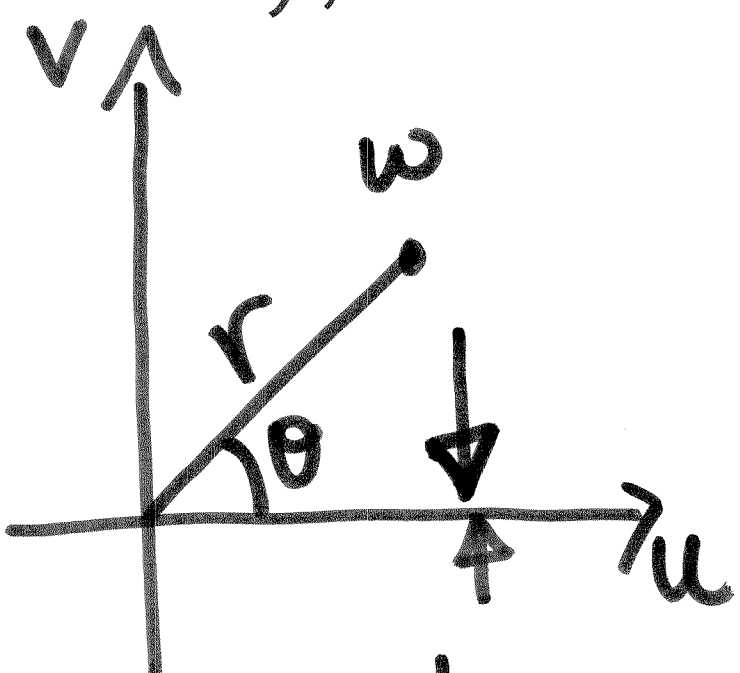
$$\begin{aligned} \text{Arg } w &= \pi + \frac{\pi}{3} \\ &= \frac{4\pi}{3} \end{aligned}$$

$$\tan^{-1} \sqrt{3} = \frac{\pi}{3}$$

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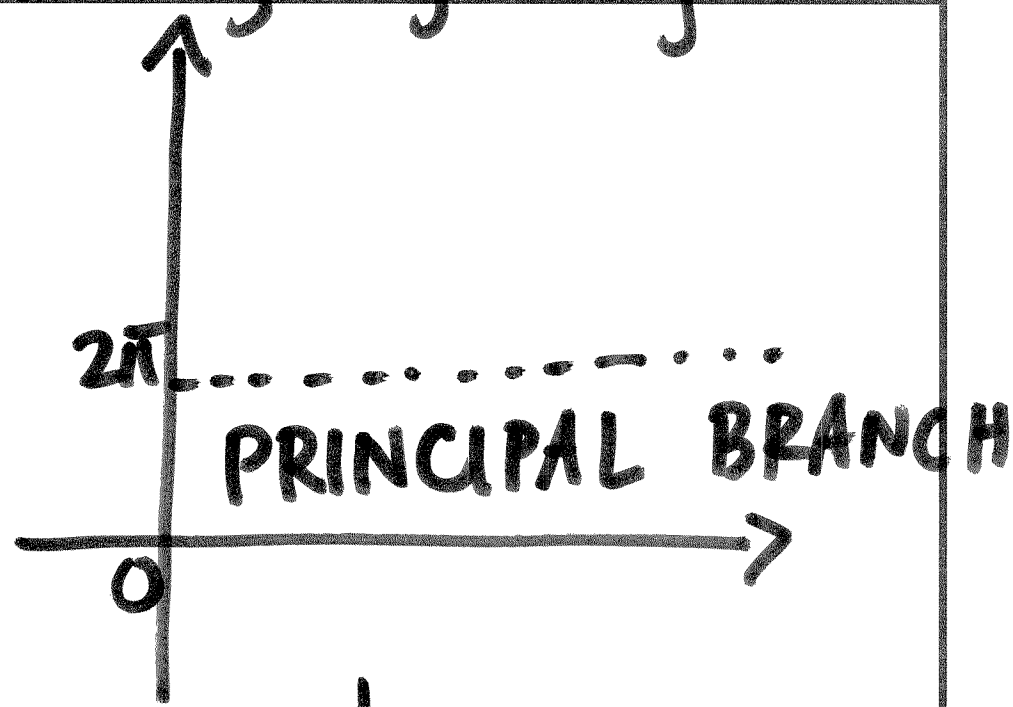
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Continuity of $\log w$



w-plane

$$w = u + iv = re^{i\theta}$$



$\log w$

$$\lim_{\substack{v \rightarrow 0^+ \\ u \geq 0}} \text{Arg } w = 0 \neq \lim_{\substack{v \rightarrow 0^- \\ u \geq 0}} \text{Arg } w = 2\pi$$

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$$\text{Log } w = \log |w| + i \text{Arg } w$$

$$\lim_{\substack{v \rightarrow 0^+ \\ u \geq 0}} \text{Log } w = \log |w| \quad \uparrow \quad \text{NOT same}$$

$$\lim_{\substack{v \rightarrow 0^- \\ u \geq 0}} \text{Log } w = \log |w| + i 2\pi$$

$\Rightarrow \lim_{w \rightarrow \bullet \text{u axis}} \text{Log } w$ does not exist

($v \rightarrow 0, u > 0$)

$\Rightarrow \lim_{v \rightarrow 0} \text{Log } w$ does not exist

$u \geq 0$

($w \rightarrow$ positive real axis)

$\Rightarrow \text{Log } w$ is not continuous along positive u axis

$\Rightarrow \text{log } w$ is not continuous along $u \geq 0$

$w = \sqrt{z}$: multivalued

$$z = r(\cos \theta + i \sin \theta) = r e^{i\theta}$$

$$w = \sqrt{z} = \sqrt{r} \left[\cos \frac{\theta + 2k\pi}{2} + i \sin \frac{\theta + 2k\pi}{2} \right]$$

$k = 0, 1$

$$= \begin{cases} \sqrt{r} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) \\ -\sqrt{r} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) \end{cases}$$

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$$z_1 = r_1 e^{i\pi/2} = r_1 e^{i(\frac{\pi}{2} + 2\pi)}$$

$$= r_1 e^{i\frac{5\pi}{2}} = z_2$$

$$\sqrt{z_1} = \sqrt{r_1} e^{i\pi/4}, \quad \sqrt{z_2} = \sqrt{r_1} e^{i5\pi/4}$$

