MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 3

Convergence of seq.s

$S = \{a_n\}_{n=1}^{\infty}$: as n gets large do the a_n's approach some number a?

The no. a is said to be the limit of S if $a_n - a$ is small for all large n

Limit of a sequence

Let $\{a_n\}$ be a sequence.

We say that the seq. approaches the limit a if for <u>every</u> $\varepsilon > 0$

$\exists N \in \mathbb{N}$ s.t.

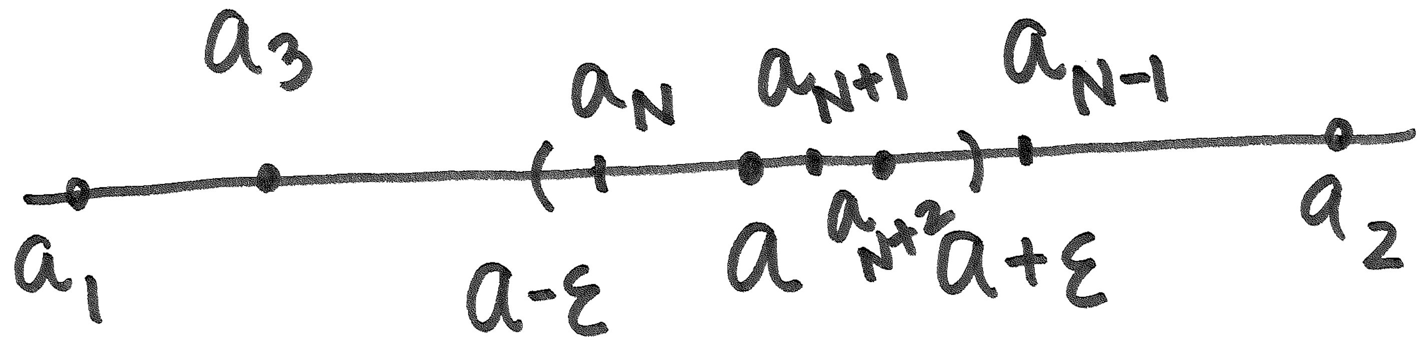
$$-\varepsilon < a_n - a < \varepsilon, \quad n \geq N$$

$$-\varepsilon < a_n - a < \varepsilon$$

or, $$|a_n - a| < \varepsilon, \quad n \geq N$$

or, $a - \varepsilon < a_n < a + \varepsilon$, $n \geq N$

or $a_n \in (a-\varepsilon, a+\varepsilon)$, $n \geq N$



At most a finite of elements $a_1, \dots, a_{N-1}$ can be outside $(a-\varepsilon, a+\varepsilon)$.

Ex 1: $\left\{ \frac{1}{n} \right\}_{n=1}^{\infty} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}$

Guess: limit is 0.

Proof: Given $\varepsilon > 0$, need to find $N \in \mathbb{N}$ s.t.

$$\left| \frac{1}{n} - 0 \right| < \varepsilon, \quad n \geq N$$

or, $\frac{1}{n} < \varepsilon, \quad n \geq N$

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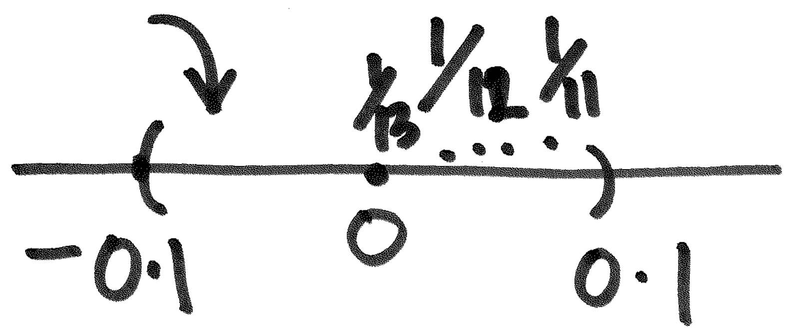
Take $N > 1/\varepsilon$. or $1/N < \varepsilon$

Then $\dfrac{1}{n} \le 1/N < \varepsilon$

$\square$

$\varepsilon = 0.1$ $\quad N > 1/0.1 = 10$

$N = 11$



$\varepsilon = 0.01$ $\quad N > 1/.01 = 100$ $\quad N = 101$

Example 2

$\{a_n\} = \{n/_{n+2}\}_{n=1}^{\infty}$ $\qquad \dfrac{n}{n+2} = \dfrac{1}{1+2/n}$

Guess : limit $a = 1$.

Proof: Given $\varepsilon > 0$, find $N \in \mathbb{N}$

s.t $\left|\dfrac{n}{n+2} - 1\right| < \varepsilon$, $n \geq N$

or, $\left|\dfrac{-2}{n+2}\right| < \varepsilon$, $n \geq N$

or, $\dfrac{2}{n+2} < \varepsilon$, $n \geq N$.

$n+2 > 2/\varepsilon$ or, $n > \dfrac{2}{\varepsilon} - 2$

Take $N > 2/\varepsilon - 2$

$\square$

$\varepsilon = 0.01$

$N > \dfrac{2}{0.01} - 2 = 198$

Let $N = 199$

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Example 3

$$\{a_n\}_{n=1}^{\infty} = \{1, 2, 3, \ldots, n, \ldots\}$$

Guess : ? a

Proof by contradiction.

Assume that $\exists$ a limit a.

Then for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$

s.t.

$$|n - a| < \varepsilon, \; n \geq N$$

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Take $\varepsilon = 1$.

$$|n - a| < 1, \quad n \geq N$$

$$-1 < n - a < 1$$

$$-1 + a < n < 1 + a, \quad n \geq N$$

i.e. for all $n \geq N$, the value of n is in $(-1+a, 1+a)$ which is not possible.

Therefore $\{a_n\}$ has <u>no</u> limit!