MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 5

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Today

- Operations on sequences
- Monotone convergence Theorem

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Sum of sequences :

$$\{a_n\}_{n=1}^{\infty} \pm \{b_n\}_{n=1}^{\infty} = \{a_n \pm b_n\}_{n=1}^{\infty}$$

$$\{1/n\}_{n=1}^{\infty} \quad \{n\}_{n=1}^{\infty} = \left\{\frac{1}{n} + n\right\}$$

$$= \{1+1, 2+1/2, \ldots$$

$$= \{2, 5/2, 3+1/3, 4+1/4, \ldots\}$$

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Product of sequences

$$\{a_n\}_{n=1}^{\infty} \cdot \{b_n\}_{n=1}^{\infty} = \{a_n b_n\}_{n=1}^{\infty}$$

$$\{1/n\} \cdot \{n\} = \{1/n \cdot n\} = \{1\}_{n=1}^{\infty}$$

$$\{1, 1, 1, \cdots\}$$

Sum of divergent seqs.

$S_1 = \{1, -1, 1, -1, \dots\}$ divergent

$S_2 = \{-1, 1, -1, 1, \dots\}$ divergent

$S_1 + S_2 = \{0, 0, 0, \dots\}$ convergent limit $= 0$

$S_3 = \{1, -1, 1, -1, \dots\}$

$S_1 + S_3 = \{2, -2, 2, -2, \dots\}$ divergent

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The sum of two divergent sequences may or may not be divergent.

$S_1 = \{n\} = \{1, 2, \ldots\}$ divergent

$S_2 = \{n + \sqrt{n}\} = \{2, 2+\sqrt{2}, 3+\sqrt{3}, \ldots\}$ divergent

$S_1 + S_2 = \{2n + \sqrt{n}\}$ divergent

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Theorem

If $\{a_n\} \to A$ and $\{b_n\} \to B$ are two convergent sequences then

a) $\lim\limits_{n\to\infty} \{a_n \pm b_n\} = A \pm B$

b) $\lim\limits_{n\to\infty} \{a_n b_n\} = AB$

c) $\lim\limits_{n\to\infty} \left\{ \frac{a_n}{b_n} \right\} = \frac{A}{B}$; $B \neq 0$

d) $\lim_{n\to\infty} \{a_n\}^p = A^p$

$$\{a_n\}^p := \{a_n^p\}$$

Ex

$$\{a_n\} = \{1 + 1/n^2\} \longrightarrow 1$$

$$\{b_n\} = \{2 + \frac{1}{2^n}\} \longrightarrow 2$$

$$\{a_n b_n\} = \left\{\left(1+\frac{1}{n^2}\right)\left(2+\frac{1}{2^n}\right)\right\} \longrightarrow 2$$

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Monotone Sequences

Def: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

1) If $a_{n+1} \geq a_n$ then $\{a_n\}$ is a monotonically increasing sequence

2) If $a_{n+1} \leq a_n$ then it is a monotonically decreasing seq.

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Example: $\{1, 2, 3, \cdots\}$

is monotonically increasing sequence

$\left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots\right\}$ is a monotonically decreasing sequence.

$\xrightarrow{}$ bounded

$|a_n| \leq \frac{1}{2}$

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$a_2 < a_1$ $a_3 = a_2$ $a_4 < a_3$

$$\left\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{5} \cdots \right\}$$

$$a_{n+1} \leq a_n$$

monotonically decreasing sequence

<u>Def</u>: A sequence that is either monotonically increasing or decreasing is called a <u>monotone</u> sequence.

$\{2, 2, 2, \ldots\}$

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Thm (Monotone Convergence Theorem)

A monotone sequence converges if and only if it is bounded.

Example $\{a_n\} = \{\sqrt{n+1} - \sqrt{n}\}$

Rationalize

$$\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} = a_n$$

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$$a_{n+1} = \frac{1}{\sqrt{n+2} + \sqrt{n+1}} < \frac{1}{\sqrt{n+1} + \sqrt{n}} = a_n$$

monotonically decreasing seq.

$\left\{ \frac{1}{\sqrt{n+1} + \sqrt{n}} \right\}$ is bounded

By MCT it is convergent.