

MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 6

Monotone Conv. Thm. (MCT)

A monotone sequence converges if & only if it is bounded.

A bounded monotone seq $\{a_n\}$

converges to

(i) $\sup \{a_n\}$ if increasing

(ii) $\inf \{a_n\}$ if decreasing

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a) (\Rightarrow) A monotone convergent sequence is bounded.
(Already shown in a previous lecture)

b) (\Leftarrow) Need to prove: a bounded monotone seq. is convergent.
We will just prove for a monotonically increasing sequence

From our assumptions, the set
 $A = \{a_1, a_2, \dots, a_n, \dots\}$ is
 bounded above & has a l.u.b.

Let $l = \text{l.u.b. } A$

Given $\varepsilon > 0$, $l - \varepsilon$ is not an upper
 bound of A . Then $\exists N \in \mathbb{N}$ s.t.

$$a_N > l - \varepsilon$$

$$a_n \geq a_N > l - \varepsilon, \quad n \geq N \quad (1)$$

(because $\{a_n\}$ is increasing)

Since l is an upper bound

$$a_n \leq l, \quad n \in \mathbb{N} \quad (2)$$

Therefore, (by (1) & (2))

$$l - \varepsilon \stackrel{(1)}{<} a_n \stackrel{(2)}{\leq} l < l + \varepsilon, \quad n \geq N$$

$$\text{or, } l - \varepsilon < a_n < l + \varepsilon, \quad n \geq N$$

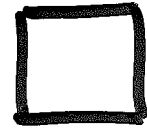
$$\text{or, } -\varepsilon < a_n - l < \varepsilon, \quad n \geq N$$

$$\text{or, } |a_n - l| < \varepsilon, \quad n \geq N$$

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Therefore, $\{a_n\}$ converges
to its l.u.b l .



Proof for monotonically
decreasing is similar.

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Revisit example (from last class)

$$\{a_n\} = \{\sqrt{n+1} - \sqrt{n}\}_{n=1}^{\infty}$$

$$a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \quad (\text{last class})$$

$\left\{ \frac{1}{\sqrt{n+1} + \sqrt{n}} \right\}$ decreasing; bounded
(last class)

by MCT, $\{a_n\}$ converges.

g. l. b of $\left\{ \frac{1}{\sqrt{n+1} + \sqrt{n}} \right\}$ is

0 and so

$$\{a_n\} \rightarrow 0.$$

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If a real number, r , satisfies $|r| < 1$, then

$$\lim_{n \rightarrow \infty} r^n = 0.$$

Proof: $r = 0$ then $\{r^n\}$

is a seq. where each element equals 0, $\lim_{n \rightarrow \infty} r^n = 0$

Let $|r| \neq 0$. Since $|r| < 1$,
we can write

$$|r| = \frac{1}{1+b}, \quad b > 0.$$

$$(1+b)^n \stackrel{\text{Binomial Thm.}}{=} 1 + nb + \frac{n(n+1)}{2} b^2 + \dots + b^n$$

$$> nb$$

$$|r^n| = \frac{1}{(1+b)^n} < \frac{1}{bn}$$

Want to
show

$\lim_{n \rightarrow \infty} r^n = 0$. Given $\varepsilon > 0$,

find $N \in \mathbb{N}$ s.t.

$$|r^n - 0| < \varepsilon, \quad n \geq N$$

$$\text{or, } |r^n| < \varepsilon, \quad n \geq N.$$

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Pick N s.t.

$$|r^n| < \frac{1}{bn} < \varepsilon, n \geq N$$

Solve for n

$$N > \frac{1}{b\varepsilon}$$

is a

choice of N that works.