MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 6

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Monotone Conv. Thm. (MCT)

A monotone sequence converges if & only if it is bounded.

$\Longleftrightarrow$

A bounded montone seq $\{a_n\}$ converges to

(i) $\sup \{a_n\}$ if increasing

(ii) $\inf \{a_n\}$ if decreasing

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Proof of MCT

a) ($\Rightarrow$) A monotone convergent sequence is bounded. (Already shown in a previous lecture)

b) ($\Leftarrow$) Need to prove: a bounded monotone seq. is Convergent. We will just prove for a monotonically increasing sequence

3

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From our assumptions, the set

$A = \{a_1, a_2, \dots, a_n, \dots\}$ is

bounded above & has a l.u.b.

Let $l = \text{l.u.b. } A$

Given $\varepsilon > 0$, $l - \varepsilon$ is not an upper bound of A. Then $\exists N \in \mathbb{N}$ s.t.

$$a_N > l - \varepsilon$$

$$a_n \geq a_N > l - \varepsilon, \quad n \geq N \qquad (1)$$

(because $\{a_n\}$ is increasing)

4

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Since ℓ is an upper bound

$$a_n \leq \ell, \quad n \in \mathbb{N} \qquad (2)$$

Therefore, (by (1) & (2))

$$\ell - \varepsilon \overset{(1)}{<} a_n \overset{(2)}{\leq} \ell < \ell + \varepsilon, \quad n \geq N$$

or, $\ell - \varepsilon < a_n < \ell + \varepsilon$, $n \geq N$

or, $-\varepsilon < a_n - \ell < \varepsilon$, $n \geq N$

or, $|a_n - \ell| < \varepsilon$, $n \geq N$

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Therefore, $\{a_n\}$ converges to its l.u.b l. $\square$

Proof for monotonically decreasing is similar.

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Revisit example (from last class)

$$\{a_n\} = \{\sqrt{n+1} - \sqrt{n}\}_{n=1}^{\infty}$$

$$a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$ (last class)

$$\left\{\frac{1}{\sqrt{n+1} + \sqrt{n}}\right\}$$ decreasing; bounded (last class)

by MCT, $\{a_n\}$ converges.

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g.l.b of $\left\{\frac{1}{\sqrt{n+1}+\sqrt{n}}\right\}$ is

0 and so

$\{a_n\} \longrightarrow 0$.

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Theorem

If a real number, r, satisfies $|r| < 1$, then $\lim_{n \to \infty} r^n = 0$.

Proof: $r = 0$ then $\{r^n\}$ is a seq. where each element equals 0, $\lim_{n \to \infty} r^n = 0$

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Let $|r| \neq 0$. Since $|r| < 1$,

we can write

$$|r| = \frac{1}{1+b}, \quad b > 0.$$

Binomial Thm.

$$(1+b)^n = 1 + nb + \frac{n(n+1)}{2} b^2 + \dots + b^n$$

$$> bn$$

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$$|r^n| = \frac{1}{(1+b)^n} < \frac{1}{bn}$$

Want to show $\lim_{n\to\infty} r^n = 0$. Given $\varepsilon > 0$,

find $N \in \mathbb{N}$ s.t.

$$|r^n - 0| < \varepsilon, \quad n \geq N$$

or, $|r^n| < \varepsilon$, $n \geq N$.

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Pick N s.t.

$$|r^n| < \frac{1}{bn} < \varepsilon, \quad n \geq N$$

Solve for n

$N > 1/b\varepsilon$ is a choice of N that works.