MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 7

Theorem (from last class)

If $r \in \mathbb{R}$ is such that

$|r| < 1$ then $\lim\limits_{n \to \infty} r^n = 0$

$r = 1$ then $\lim\limits_{n \to \infty} 1^n = 1.$

$|r| > 1$ then $\{r^n\}$ diverges.

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Thm proved earlier

Convergence $\Longrightarrow$ boundedness

not bounded $\Longrightarrow$ not convergent

If $|r|>1$ then $\{r^n\}$ is unbounded and therefore divergent

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Comparison Test

If $\{a_n\}_{n=1}^{\infty}$ is a sequence that diverges to $+\infty$ and $a_n \leq b_n$ for all $n \geq n_1$, then $\{b_n\}_{n=1}^{\infty}$ also diverges to $+\infty$.

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$$a_n = \left\{ n + \left(-\frac{1}{n}\right)^n \right\} \longrightarrow \infty$$

$\{a_n\}$ diverges to $+\infty$ if for any $M \in \mathbb{R}$ $\exists N \in \mathbb{N}$ such that

$$a_n > M \quad \text{for all } n \geq N$$

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$$\{a_n\}_{n=1}^{\infty} = \{n^2\}_{n=1}^{\infty} \xrightarrow{\text{diverges}} \infty$$

$$\{b_n\}_{n=1}^{\infty} = \{n^2 + 1/n\}_{n=1}^{\infty}$$

$$\underset{a_n}{n^2} < \underset{b_n}{n^2 + 1/n}$$

By the Comparison Test $\{b_n\}$ also diverges.

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Sandwich or Squeeze Thm.

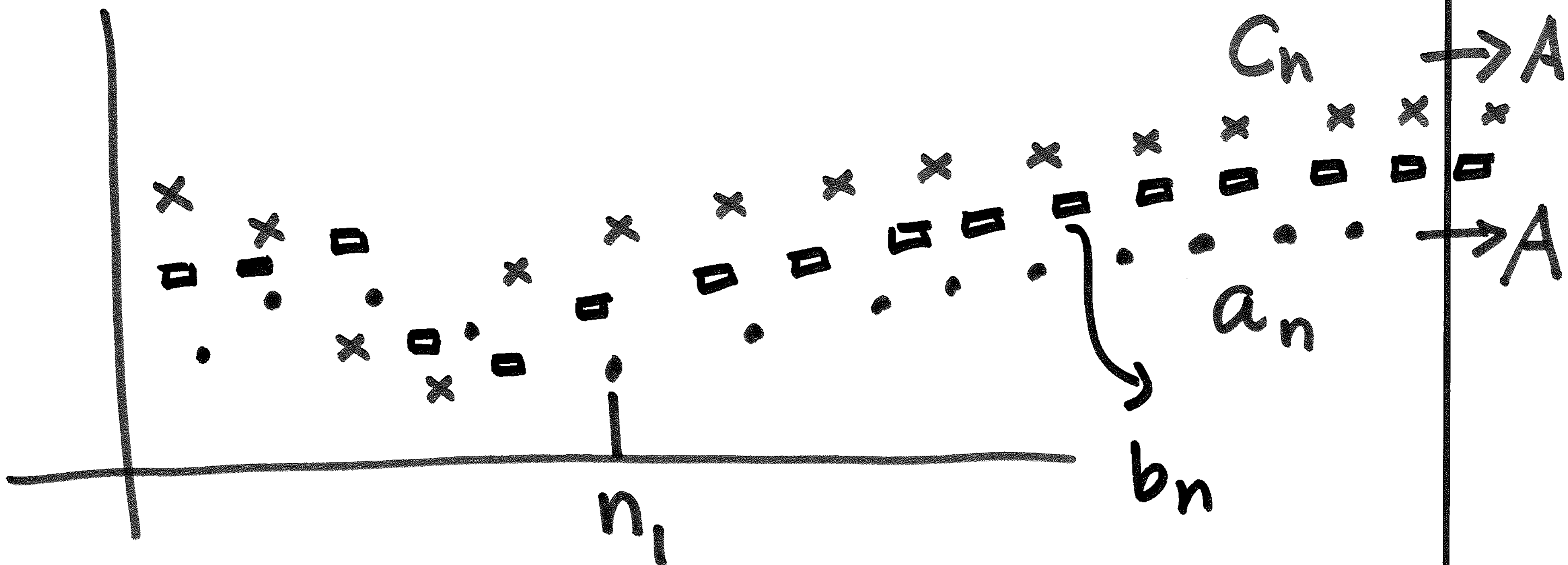
Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences and for some $n_1 \in \mathbb{N}$

$$a_n \le b_n \le c_n \quad \text{for all } n \ge n_1$$

If $\lim_{n \to \infty} a_n = A$ & $\lim_{n \to \infty} c_n = A$

then

$$\lim_{n \to \infty} b_n = A.$$



Ex. $\left\{ \frac{\sin n}{n} \right\}_{n=1}^{\infty}$ Let $b_n = \frac{\sin n}{n}$

$a_n = -\frac{1}{n}$, $c_n = 1/n$

$$-\underbrace{\frac{1}{n}}_{a_n} \le \underbrace{\frac{\sin n}{n}}_{b_n} \le \underbrace{\frac{1}{n}}_{c_n}$$

Since $\lim\limits_{n\to\infty}\left\{\frac{-1}{n}\right\} = \lim\limits_{n\to\infty}\left\{\frac{1}{n}\right\}$

$= 0$

we have (by the Squeeze Thm) that $\lim\limits_{n\to\infty} \frac{\sin n}{n} = 0$.

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Summary :

Convergence & limits of seqs

- Prove convergence using the definition
- Convergent seqs are bounded (converse is not true)
- Monotone Convergence Theorem
- Comparison Test (divergence)
- Squeeze Thm (convergence)

Subsequences

A subsequence is obtained from a given sequence $\{a_n\}$ by omitting some terms of $\{a_n\}$ without changing the order.

<u>Example</u>: $\{a_n\} = \{n\}_{n=1}^{\infty}$, $\{b_n\} = \{2n\}_{n=1}^{\infty}$

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$\{a_n\} = \{1, 2, 3, 4, 5, 6, \dots\}$

$\{b_n\} = \{2, 4, 6, \dots\}$

$\{b_n\}$ is a subsequence of $\{a_n\}$

$a_2 = b_1, \; a_4 = b_2, \dots, a_{2n} = b_n$

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Example

$$a_n = 1/n \;, \quad b_n = \frac{1}{n^2}$$

$$\{a_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$$

$$\{b_n\} = \left\{1, \frac{1}{4}, \frac{1}{9}, \ldots\right\}$$

$\{b_n\}$ is a subsequence of $\{a_n\}$; $\quad a_{n^2} = b_n$

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$$\{a_n\} = \{2n+1\}_{n=1}^{\infty}, \quad \{b_n\} = \{n\}_{n=1}^{\infty}$$

$$\{a_n\} = \{3, 5, 7, \cdots\}$$

$$\{b_n\} = \{1, 2, 3, \cdots\}$$

b_n is <u>not</u> a subsequence of a_n but $\{a_n\}$ is a subsequence of $\{b_n\}$.

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Every sequence is a subsequence of itself.