MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 8

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Subsequences

Def Given $\{a_n\}_{n=1}^{\infty}$. Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of naturals s.t.

$$n_1 < n_2 < n_3 < \cdots$$

Then $\{b_k\}_{k=1}^{\infty}$ given by

$$b_k = a_{n_k} \text{ for all } k$$

is a subsequence of $\{a_n\}_{n=1}^{\infty}$.

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$a_1 \quad a_2 \quad a_3 \quad \ldots$

$b_1 = a_3$ $\qquad n_1 = 3$

$b_2 = a_7$ $\qquad n_2 = 7$

$b_3 = a_{10}$ $\qquad n_3 = 10$

$\vdots$

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Theorem

A sequence converges to a if and only if every subsequence converges to a.

Remark: Cannot be used as a test for convergence.

Can be used to test for divergence

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$\{1, -1, 1, -1, \cdots\}$

$\{1, 1, 1, \cdots\} \longrightarrow 1$

convergent subsequence

$\{-1, -1, -1, \cdots\} \longrightarrow -1$

Convergent subsequence

$1 \neq -1$, the seq is divergent.

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Proof of Theorem:

($\Leftarrow$) Assume that every subsequence converges to a. Since the sequence is a subsequence of itself, it also converges to a by assumption

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($\Rightarrow$) Suppose that $\{a_n\}$ converges to a. Need to show that every subsequence converges to a.

Let $\{a_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{a_n\}_{n=1}^{\infty}$. Given $\varepsilon > 0$, need to find $N \in \mathbb{N}$ s.t.

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$$|a_{n_k} - a| < \varepsilon, \quad n_k \geqslant N.$$

Since $a_n \to a$, $\exists\, N^* \in \mathbb{N}$ s.t.

$$|a_n - a| < \varepsilon, \quad n \geqslant N^*.$$

Since $\{a_{n_k}\}$ is a subsequence we should have

$$|a_{n_k} - a| < \varepsilon, \quad n_k \geqslant N^*.$$

Take $N = N^*$.

Therefore $\{a_{n_k}\}$ converges to a. $\square$

$a_1 \; a_2 \; a_3 \cdots a_{n_1} \cdots a_{n_2} \cdots a_{N^*} \cdots a_{n_k} \; a_{n_{k+1}} \cdots$

known

$|a_n - a| < \varepsilon$

$|a_{n_k} - a| < \varepsilon$

$N = N^*$

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$\{1, -1, 1, -1, 1, -1, \dots\}$

bounded, divergent but it has convergent subsequences

$\{1, 1, \dots\}$

$\{-1, -1, \dots\}$

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Every bounded sequence has a convergent subsequence.

Bolzano-Weierstrass Theorem

(Thm 2.33 Fitzpatrick)