MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 9

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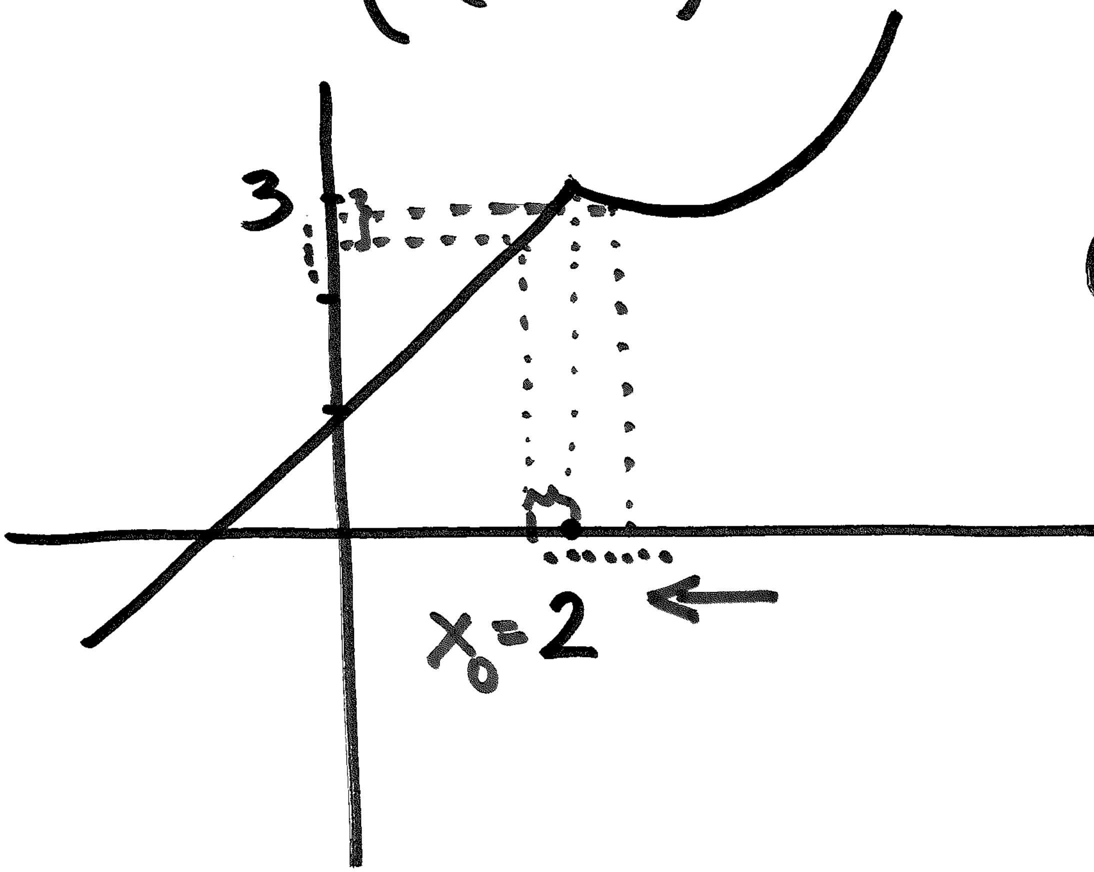
Continuity

$$f(x) = \begin{cases} x+1 & x \le 2 \\ (x-2)^2+3 & x > 2 \end{cases}$$



Continuous

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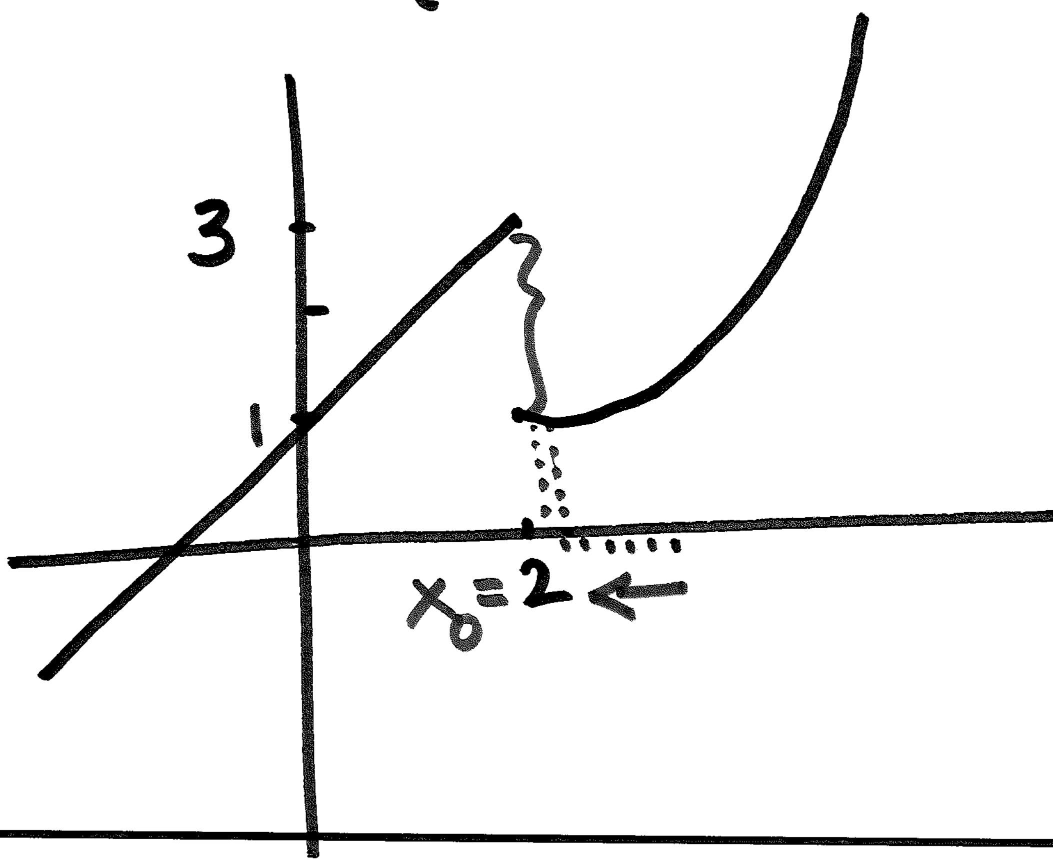
$$f(x) = \begin{cases} x+1 & x \le 2 \\ (x-2)^2+1 & x > 2 \end{cases}$$



not continuous at $x=2$.

Intuition:

For a function to be continuous at $x = x_0$: whenever x is close x_0, $f(x)$ is also close to $f(x_0)$.

For a discontinuous function even if x is close to x_0, $f(x)$ need not be close to $f(x_0)$.

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$$f : D \longrightarrow \mathbb{R}, \quad D \subseteq \mathbb{R}$$
$$x \longmapsto f(x)$$

A function f is continuous at $x_0 \in D$ if whenever a sequence $\{x_n\}$ converges to x_0 we have $\{f(x_n)\}$ converges to $f(x_0)$.

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Example

$f(x) = x^2 - 2x + 4$; $f : \mathbb{R} \to \mathbb{R}$

Pick $x_0 \in \mathbb{R}$. Let

$\{x_n\} \to x_0$.

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}(x_n^2 - 2x_n + 4)$$

$$= x_0^2 - 2x_0 + 4$$

$$= f(x_0)$$

f is continuous at x_0.

Example

$k \in \mathbb{N}$

Any polynomial of degree k is continuous : $\{x_n\} \longrightarrow x_0$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$$

$$f(x_n) = a_0 + a_1 x_n + \cdots + a_k x_n^k$$

$$\lim_{n \to \infty} f(x_n) = a_0 + a_1 x_0 + \cdots + a_k x_0^k$$

$$= f(x_0)$$

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Trig functions like

$\sin x$, $\cos x$, $\cdots$

are continuous functions

Ex. of a discontinuous function

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Dirichlet function

This is not a continuous function.

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University of Idaho Fact: (Thm 2.20 Fitzpatrick)

For any $x_0 \in \mathbb{R}$, there is a sequence of rationals $\{r_n\}$ such $\{r_n\} \rightarrow x_0$.

$x_0 = \sqrt{2}$

$\{1, 1.4, 1.41, 1.414, \ldots\} \rightarrow \sqrt{2}$

rationals

Let x_0 be irrational. Then

$\exists$ $\underbrace{\{r_n\}}_{}$ ← rationals , $\{r_n\} \longrightarrow x_0$.

$f(r_n) = 1$, $\{f(r_n)\} \to 1$

but $f(x_0) = 0$

When x_0 is rational use the fact: For any x_0, $\exists$ a seq.

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$\{v_n\}$ of irrational nos. s.t.

$\{v_n\} \longrightarrow x_0$.

$\{f(v_n)\} = \{0\} \longrightarrow 0$

but $f(x_0) = 1 \neq 0$

Therefore, the Dirichelt function is discontinuous at every $x_0 \in \mathbb{R}$.

Properties of continuous functions ;

Suppose $f, g : D \longrightarrow \mathbb{R}$, $D \subseteq \mathbb{R}$, are continuous at a.

a) $f \pm g$ is continuous at a.

b) fg is continuous at a

c) f/g is continuous at a; $g(a) \neq 0$.

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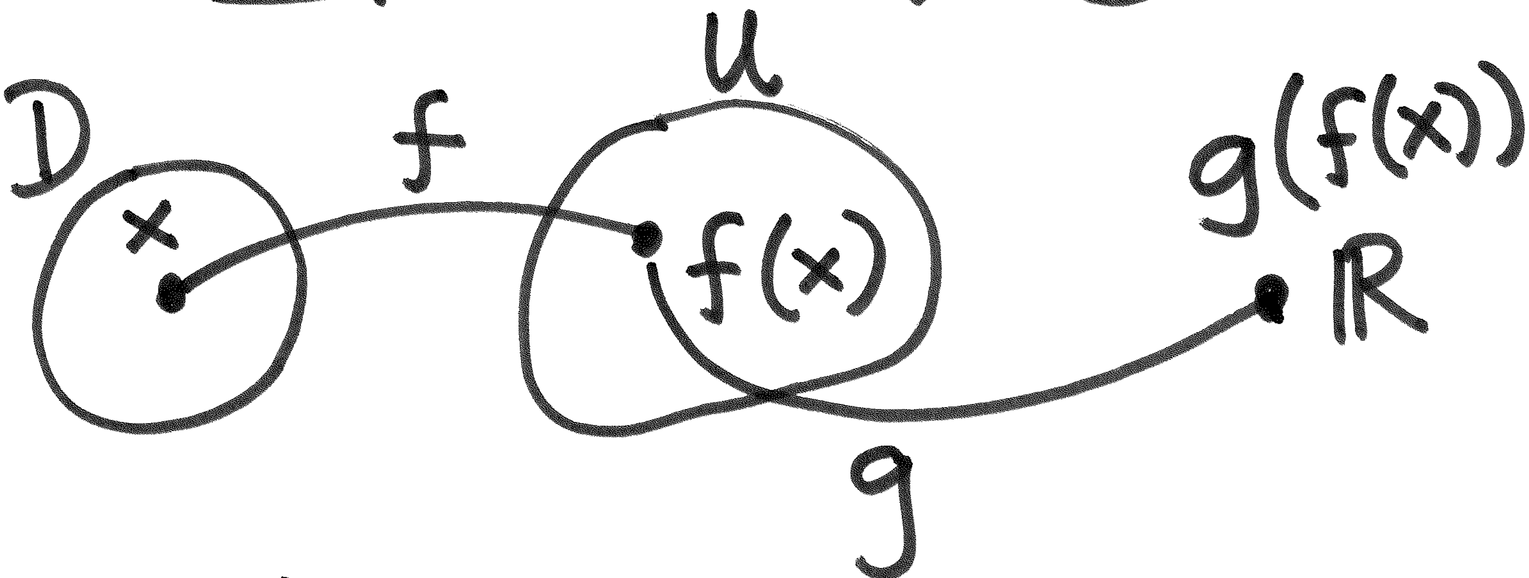
Composition of continuous functions

Def: $f : D \to \mathbb{R}$, $g : U \to \mathbb{R}$ such that $f(D) \subseteq U$. Then the composition of f & g denoted by $g \circ f : D \to \mathbb{R}$ is defined by

$$g \circ f(x) = g(f(x)), \ \forall x \in D.$$

$\boxed{D} \xrightarrow{f} \mathbb{R}, \quad U \xrightarrow{g} \mathbb{R}$



$f(D) \subseteq U$

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$f(x) = 1 + \sin x$ — continuous

$g(x) = x^2$ — continuous

$$g \circ f(x) = g(f(x))$$
$$= g(1 + \sin x)$$
$$= (1 + \sin x)^2$$ — also continuous

$$f \circ g(x) = f(x^2) = \underbrace{1 + \sin x^2}_{\text{continuous}}$$

Theorem

If f & g are continuous,

then

$$g \circ f$$

is also continuous.

[Proof in next lecture].