MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 11

1

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Today:

Proof of the Extreme Value Theorem

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Recall

Bolzano-Weierstrass Thm:

Every bounded sequence must have a convergent subsequence.

$\{1, -1, 1, -1, \ldots\}$ bounded divergent

$\{1, 1, 1, \ldots\}$ convergent

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Recall

A function f is bounded on $[a, b]$ if $\exists\, M \in \mathbb{R}$ such that

$$|f(x)| \leq M, \quad \forall x \in [a,b].$$

Theorem

If a function f is continuous on a closed & bounded interval $[a, b]$, then f is bounded on $[a, b]$.

<u>Proof</u>: Proceed by contradiction: Assume that f is not bounded.

5

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Then there exists a seq

$\{x_n\}_{n=1}^{\infty} \in [a,b]$ s.t. $|f(x_n)| > n$ bdd.

for all $n \in \mathbb{N}$. There is

a convergent subsequence,

$\{x_{n_k}\}_{k=1}^{\infty} \in [a,b]$. (By the Bolzano. W Thm).

$$\lim_{k \to \infty} x_{n_k} = c \in [a,b].$$

$\{f_{\cancel{n_k}}(x_{n_k})\}$

$\lim\limits_{k \to \infty} f(x_{n_k}) = f(c)$; since f is continuous. This contradicts the assumption that f is not bounded.

Therefore, f must be bounded on $[a, b]$ $\square$

Extreme Value Thm.

If f is continuous on $[a, b]$ (closed & bounded), then f attains both max and min on $[a, b]$.

Proof: By the previous thm, f is bounded.

[Proof only for max; Proof for min is similar]

8

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f has a l.u.b., denoted by M.

$$M = \sup\{f(x) : a \leq x \leq b\}$$
$$= \text{l.u.b } f$$

Proceed by contradiction, assume that f does not attain a max on $[a,b]$,

9

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i.e., $\nexists$ $c \in [a,b]$ s.t.

$$f(c) = M.$$

Therefore,

$$f(x) < M \text{ for all } x \in [a,b].$$

Define a function

$$g(x) = \frac{1}{M - f(x)} > 0$$

g is continuous on $[a,b]$.

g is bounded by the prev. thm.

Then $\exists\, K \in \mathbb{R}$, such that

$$g(x) \leq K \Rightarrow \frac{1}{M - f(x)} \leq K$$

$$\Rightarrow f(x) \leq \underbrace{M - \frac{1}{K}}, \quad \forall x \in [a,b]$$

$\hookrightarrow$ seems to be an upper bdd.

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This is not possible since M is the l.u.b. and so $M - 1/K$ cannot be an upper bound. Hence f must attain a max on $[a,b]$, i.e., $\exists\, c \in [a,b]$ s.t.

$$f(c) = M.$$

$\square$