MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 12

1.

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(IVT)

Intermediate Value Thm (for continuous functions)

Let f be a continuous function on $[a,b]$. Let c be s.t.

$f(a) < c < f(b)$ or $f(b) < c < f(a)$

Then $\exists\ x_0 \in (a,b)$ s.t.

$a < x_0 < b$

$$f(x_0) = c.$$

[Thm 3.11 in Fitzpatrick]

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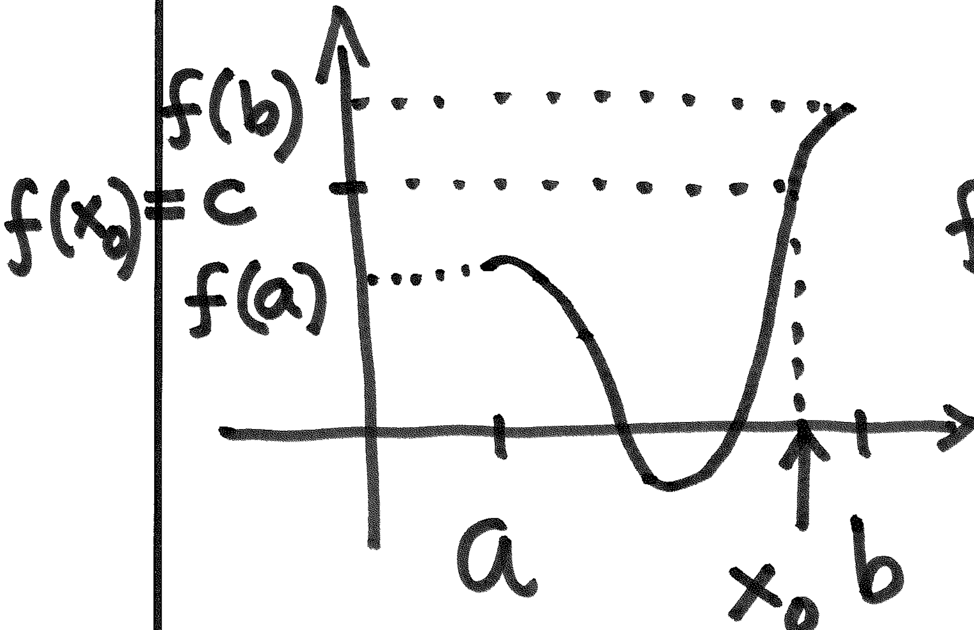
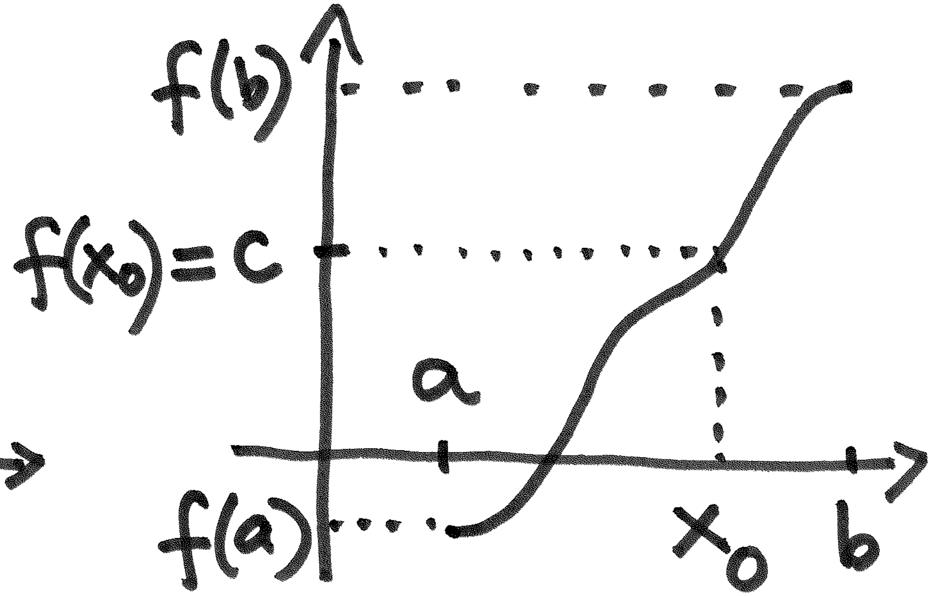
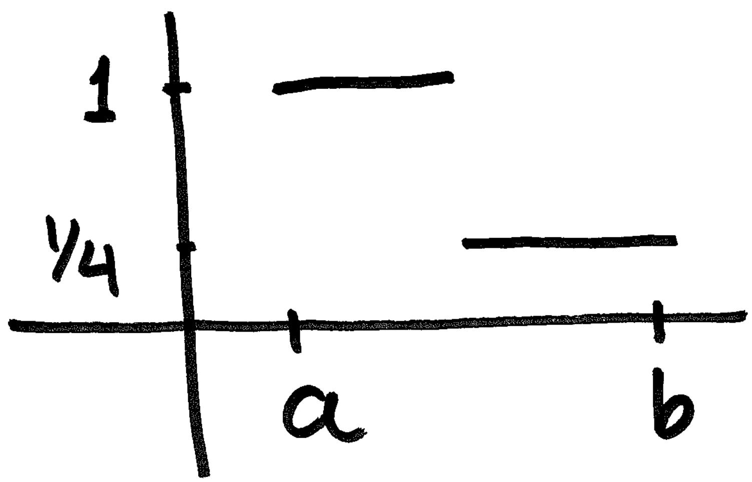
Continuous





Discontinuous



$$\underset{f(b)}{1/4} < \underset{c}{1/2} < \underset{f(a)}{1}$$

$f(a) = 1$

$f(b) = 1/4$

Let $c = 1/2$

$\nexists\, x_0 \in (a,b)$ s.t.

$f(x_0) = 1/2$

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Root :

Any solution of the equation

$p(x) = 0$

is called a root of $p(x)$.

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Applications of the IVT

1) Any polynomial of odd degree has at least one real root.

Proof: Let

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

n : odd, positive, $a_n \neq 0$

Assume that $a_n > 0$.

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$a_n > 0$

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$p(x)$ is continuous for all $x \in \mathbb{R}$.

$$\lim_{x \to \infty} p(x) = \lim_{x \to \infty} \left(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0\right)$$

$$= \lim_{x \to \infty} x^n \left(a_n + \underbrace{\frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}}_{\text{get closer to zero for large } x}\right)$$

$\Rightarrow p(x) > 0$ for large x, $x > 0$.

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Similarly, by looking at $\lim_{x \to -\infty} p(x)$

$p(x) < 0$ if x is large

(in absolute value) & negative

$\exists\ a, b \in \mathbb{R}$ s.t.

$p(a) < \underset{\hookrightarrow c}{0} < p(b)$

Since p is continuous, by the IVT

$\exists\ x_0 \in (a,b)$ s.t. $p(x_0) = 0$

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Compare with Exs 3.12 & 3.13 in Fitzpatrick

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If $p(x)$ has even degree:

$$p(x) = x^2 + 4 = 0$$

$$\Rightarrow x^2 = -4$$

Does not hold for real x.

For odd degree

$$x^3 + 8 = 0$$

$$x^3 = -8 \Rightarrow x = (-1)\sqrt[3]{8} = -2.$$

Applications of the IVT

2) If $f : [a, b] \rightarrow \mathbb{R}$ is non constant and continuous, then the range of f is a closed & bounded interval $[c, d]$.

interval

Proof: By the extreme value thm, f attains a max & a min in $[a, b]$. This means that $\exists$

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$x_1, x_2 \in [a,b]$ s.t. $\forall x \in [a,b]$

$f(x_1) \leq f(x) \leq f(x_2)$

min max

Let $c = f(x_1)$ & $d = f(x_2)$

and we know $c < d$.

By the IVT all values between $f(x_1) = c$ and $f(x_2) = d$ will be attained.

Therefore, the range of f is $[c, d]$; f maps $[a, b]$ onto $[c, d]$

$\square$

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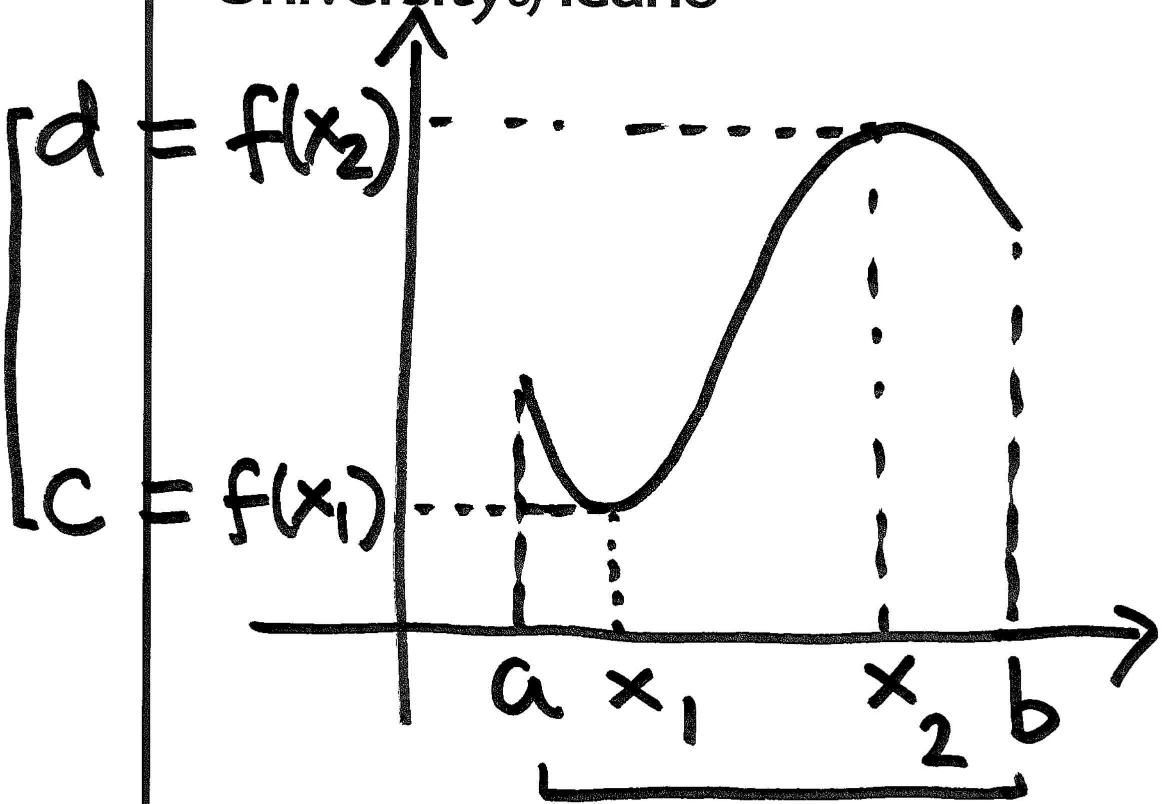

Ex: $f(x) = x$