MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 13

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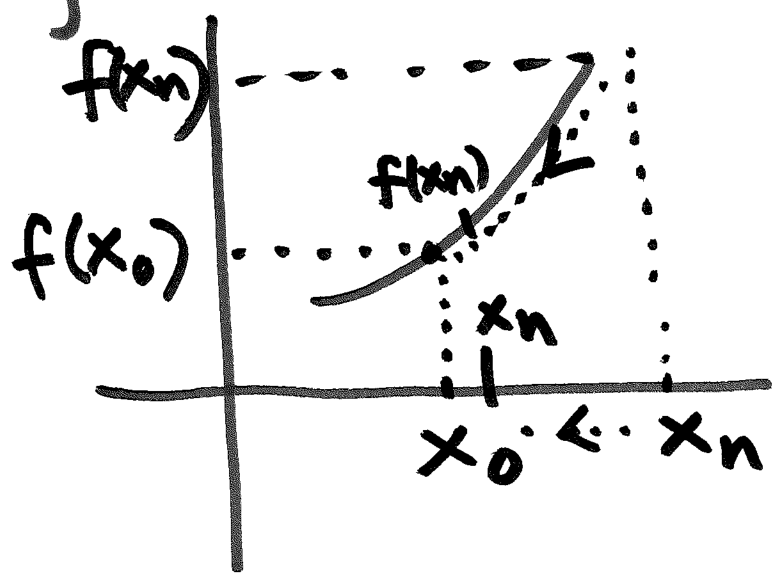
Uniform Continuity

$f : D \longrightarrow \mathbb{R}$

f is <u>continuous</u> at $x_0 \in D$ if for any seq. $\{x_n\} \to x_0$, $\{f(x_n)\} \longrightarrow f(x_0)$.



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$f: D \longrightarrow \mathbb{R}$

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Def: A function f is uniformly continuous if whenever $\{u_n\}$ and $\{v_n\}$ are sequences s.t.

$$\lim_{n\to\infty} [u_n - v_n] = 0$$

then

$$\lim_{n\to\infty} [f(u_n) - f(v_n)] = 0.$$

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Ex: 1

$f : (0,1) \longrightarrow \mathbb{R}$; $f(x) = \dfrac{1}{x}$

$(0,1) \rightarrow$ bounded

f is continuous (ratio of two continuous functions 1 & x ; $x \neq 0$ on $(0,1)$)

However, f is <u>not</u> uniformly continuous.

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Let $u_n = \frac{1}{n}$ & $v_n = \frac{1}{2n}$

$\lim_{n\to\infty} [u_n - v_n] = \lim_{n\to\infty} \frac{1}{2n} = 0$

$f(u_n) = n$; $f(v_n) = 2n$

$\lim_{n\to\infty} f(u_n) - f(v_n) = \lim_{n\to\infty} (-n) \neq 0$

So f is <u>not</u> unif. continuous.

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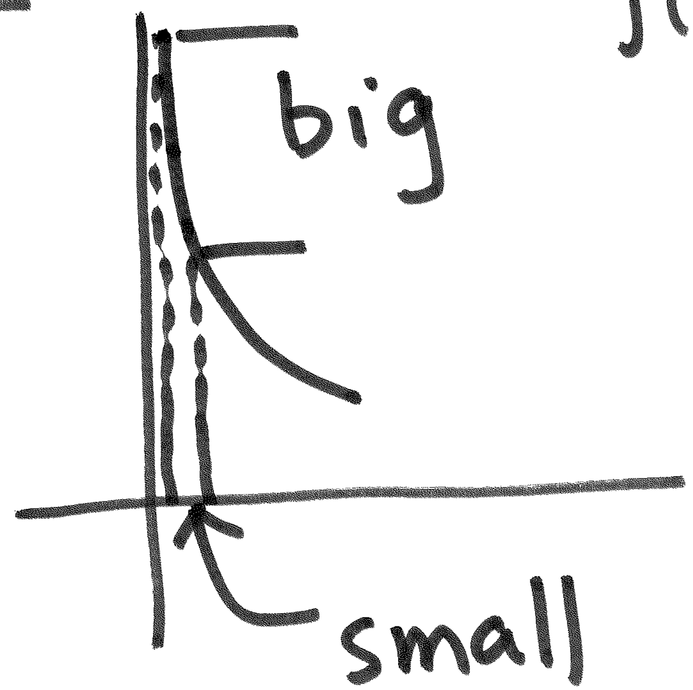
gets big as you get closer zero

$f(x) = \frac{1}{x}$



Ex #2

$f(x) = x^2$; $f : \mathbb{R} \to \mathbb{R}$ un-bounded

f is continuous ; f is **not** uniformly continuous.

$u_n = n$, $v_n = n + \frac{1}{n}$

$$\lim_{n \to \infty} [u_n - v_n] = \lim_{n \to \infty} -\frac{1}{n} = 0$$

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$f(x) = x^2$

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$$f(u_n) = f(n) = n^2$$

$$f(v_n) = f\left(n+\frac{1}{n}\right) = \left(n+\frac{1}{n}\right)^2$$

$$= n^2 + 2 + \frac{1}{n^2}$$

$$\lim_{n\to\infty}\left[f(u_n) - f(v_n)\right] = \lim_{n\to\infty}\left[-2 - \frac{1}{n^2}\right]$$

$$= -2 \neq 0$$

$\Rightarrow$ f is NOT unif. Continuous.

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Ex #3

$f(x) = x \quad ; \quad f : \mathbb{R} \longrightarrow \mathbb{R}$

Let $\{u_n\}$ & $\{v_n\}$ be any two sequences in $\mathbb{R}$ s.t.

$$\lim_{n\to\infty} [u_n - v_n] = 0$$

$$f(u_n) = u_n \quad \& \quad f(v_n) = v_n$$

$$\lim_{n\to\infty} [f(u_n) - f(v_n)] = \lim_{n\to\infty} [u_n - v_n] = 0$$

$\Rightarrow f$ is uniformly continuous.

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Remarks

1) No assumptions on the convergence of $\{u_n\}$ & $\{v_n\}$

2) Intuitive notion: Whenever any two points u & v come very close then $f(u)$ & $f(v)$ also come very close regardless of where u & v are in D.

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3) If f is unif. continuous then it is also continuous. [not conversely see Ex 1 & 2 today]

Proof: Let x_0 be any point in D. Let $\{x_n\} \to x_0$.

Set $u_n = x_n$ & $v_n = x_0$

$$\lim_{n\to\infty} [u_n - v_n] = \lim_{n\to\infty} [x_n - x_0] = 0$$

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Due to uniform continuity

$$\lim_{n \to \infty} [f(x_n) - f(x_0)] = 0$$

$$\Rightarrow \lim_{n \to \infty} f(x_n) = f(x_0)$$

$\Rightarrow$ f is continuous.

$\square$

Ex: 4

$f(x) = x^2$; $f : [0,1] \longrightarrow [0,1]$

Now $f(x)$ is uniformly continuous.

Next class: A continuous func. on a <u>closed</u> & <u>bounded</u> interval is uniformly continuous.