MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 14

1. University of Idaho **Uniform Continuity**

Theorem: If $f : D \to \mathbb{R}$; D is closed & bounded; f is continuous then f is uniformly continuous.

Proof: Proceed by contradiction. Assume that f is NOT unif. continuous.

Let $\varepsilon > 0$. Pick two sequences $\{x_n\}$ & $\{t_n\}$ in D s.t.

$|x_n - t_n| \leq \frac{1}{n}$ but $|f(x_n) - f(t_n)| \geqslant \varepsilon$.

Since $\{x_n\}$ & $\{t_n\}$ are bounded, by the Bolzano - Weierstrass thm they have convergent subsequence(s).

3 $\{t_{n_k}\}$ is a subseq. of $\{t_n\}$.

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Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ that converges to α.

α is in D because D is closed.

Since f is continuous

$$\lim_{k \to \infty} f(x_{n_k}) = f(\alpha).$$

$$|t_{n_k} - \alpha| \leq |t_{n_k} - x_{n_k}| + |x_{n_k} - \alpha|$$

by Triangle

$$|t_{n_k} - \alpha| \leq \frac{1}{n_k} + |x_{n_k} - \alpha|$$

Thus $\{t_{n_k}\}$ also converges to α.

Therefore, $\lim\limits_{k\to\infty} f(t_{n_k}) = f(\alpha)$,

Contradicting the fact

$$|f(x_n) - f(t_n)| \geqslant \varepsilon.$$

Thus f is uniformly continuous. $\square$

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Ex

closed; unbounded

1. $f(x) = x^2$ $f : \mathbb{R} \longrightarrow \mathbb{R}$

continuous

f is not uniformly cont.

Theorem cannot be used.

2. $f(x) = x$ $f : \mathbb{R} \longrightarrow \mathbb{R}$

f is uniformly cont.

Converse of the previous theorem does not hold.

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3. $f(x) = \frac{1}{x}$ $f : (0,1) \to \mathbb{R}$

continuous but not unif-cont.

bounded but not closed.

Theorem cannot be used.

4 $f(x) = x^2$ $f : [0,2] \to \mathbb{R}$

f is continuous, defined on $[0,2]$ - closed & bounded. By the Theorem f is uniformly cont.

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5 $f(x) = \dfrac{1}{x}$ $\qquad f : \underbrace{[1, 2]}_{D} \to \mathbb{R}$

or $f : \left[\dfrac{1}{\varepsilon}, 1\right] \to \mathbb{R}$

D → $\varepsilon > 1$

Then f is unif. cont. on D.

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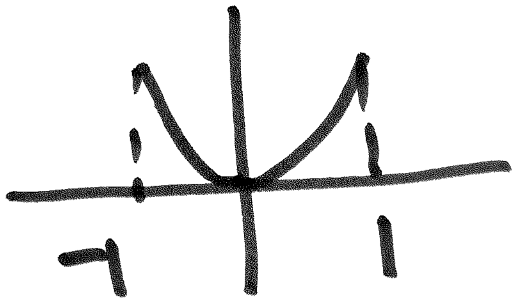
6. $f(x) = x^2$; $f: [-1,1] \to \mathbb{R}$



f is unif. cont. by the theorem.

Scratch

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$|a+b| \le |a| + |b|$

$$|t_{n_k} - \alpha|$$

$$= |\underbrace{t_{n_k} - x_{n_k}} + \underbrace{x_{n_k} - \alpha}|$$

$$\le |t_{n_k} - x_{n_k}| + |x_{n_k} - \alpha|$$