MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 15

Lipschitz functions

Def $f : D \to \mathbb{R}$ is called a Lipschitz function if $\exists\, C > 0$ such that

$$|f(u) - f(v)| \leq C\,|u - v|$$

for all $u, v \in D$.

C : Lipschitz constant

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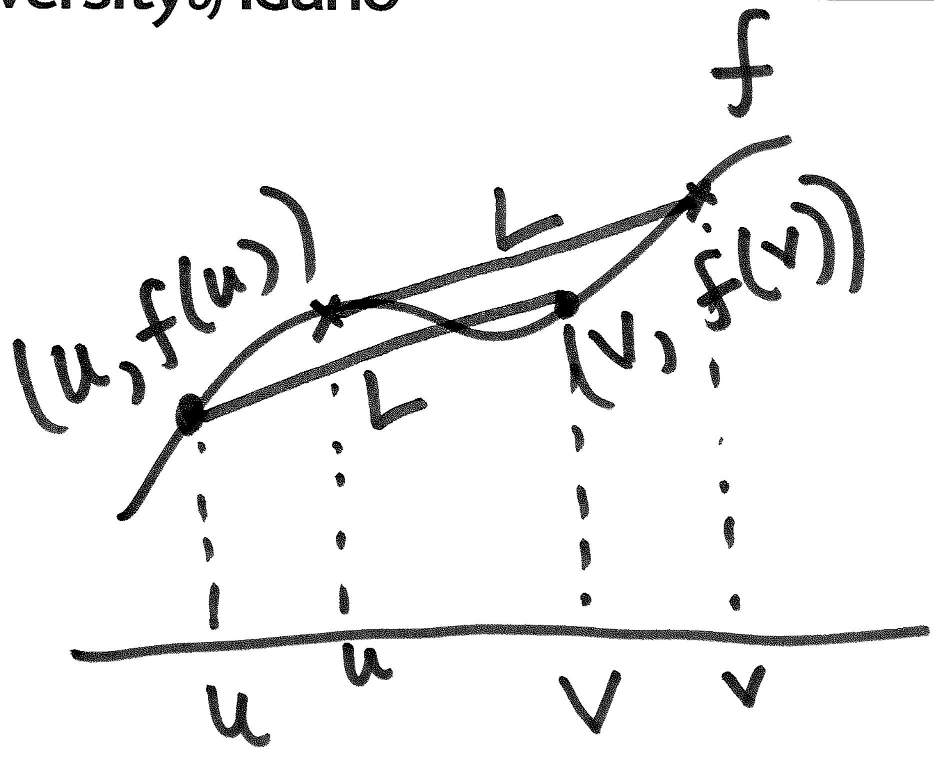
We need:

$$\frac{|f(u) - f(v)|}{|u - v|} \le C, \forall u \neq v.$$

$\Rightarrow$ The slope of the line joining $(u, f(u))$ & $(v, f(v))$ must be bounded by C.



If f is Lipschitz then $\exists\, C$ s.t. slope $L \leq C$.

Ex.1 : $f(x) = x$. $f: \mathbb{R} \to \mathbb{R}$

$$|f(u) - f(v)| = |u - v|$$

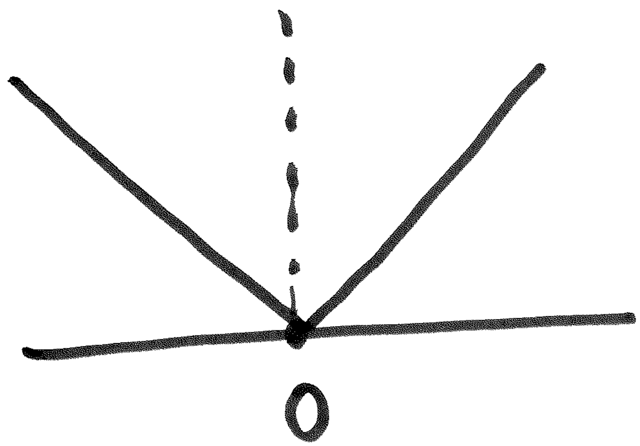
For $C = 1$, $|f(u) - f(v)| \le |u - v| \quad \forall u, v \in \mathbb{R}$

$\Rightarrow$ f is Lipschitz with Lipschitz constant $= 1$.

$$|f(u) - f(v)| = |u - v|$$
$$\leq 2|u - v|$$
$$\leq C|u - v|$$
where $C \geq 1$

Any $C \geq 1$ is a Lipschitz constant for $f(x) = x$.

Ex 2. $f(x) = |x|$; $f : \mathbb{R} \to \mathbb{R}$



f is also a Lipschitz func with Lipschitz constant $= 1$.

[Try to prove this].

$f : D \to \mathbb{R}$

Theorem (Exercise # 11 in 3.4, Fitzpatrick)

If f is Lipschitz, then it is uniformly continuous.

<u>Proof</u>: Let $\{u_n\}$ and $\{v_n\}$ be sequences in D such that

$$|u_n - v_n| \to 0, \quad n \to \infty.$$

8

Since f is Lipschitz, $\exists\, C$ s.t.

$\forall n$, $|f(u_n) - f(v_n)| \le C|u_n - v_n|$

$$\underbrace{0}_{a_n} \le \underbrace{|f(u_n) - f(v_n)|}_{b_n} \le \underbrace{C|u_n - v_n|}_{c_n}$$

Since $|u_n - v_n| \to 0$, by the Squeeze thm.

$$\lim_{n \to \infty} |f(u_n) - f(v_n)| = 0$$

$\Rightarrow f$ is uniformly continuous. $\square$

Example

$f : [0,1] \longrightarrow \mathbb{R}$; $f(x) = \sqrt{x}$

f is continuous and since $[0,1]$ is closed & bounded, f is also uniformly continuous

But f is NOT Lipschitz.

$f = \sqrt{x}$

The converse of the theorem is NOT true.

10

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$$\frac{|f(u)-f(0)|}{|u-0|} = \frac{f(u)}{u} = \frac{\sqrt{u}}{u}$$

$$= \frac{1}{\sqrt{u}}$$ gets bigger as u approaches zero. Thus

$\not\exists$ C s.t.

$$\frac{|f(u)-f(0)|}{|u-0|} = \frac{1}{\sqrt{u}} \leq C$$

$\Rightarrow$ f is not Lipschitz.

11

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Lipschitz $\implies$ uniform continuity
$\not\Longleftarrow$

uniform continuity $\overset{\not\Longleftarrow}{\implies}$ continuity

12

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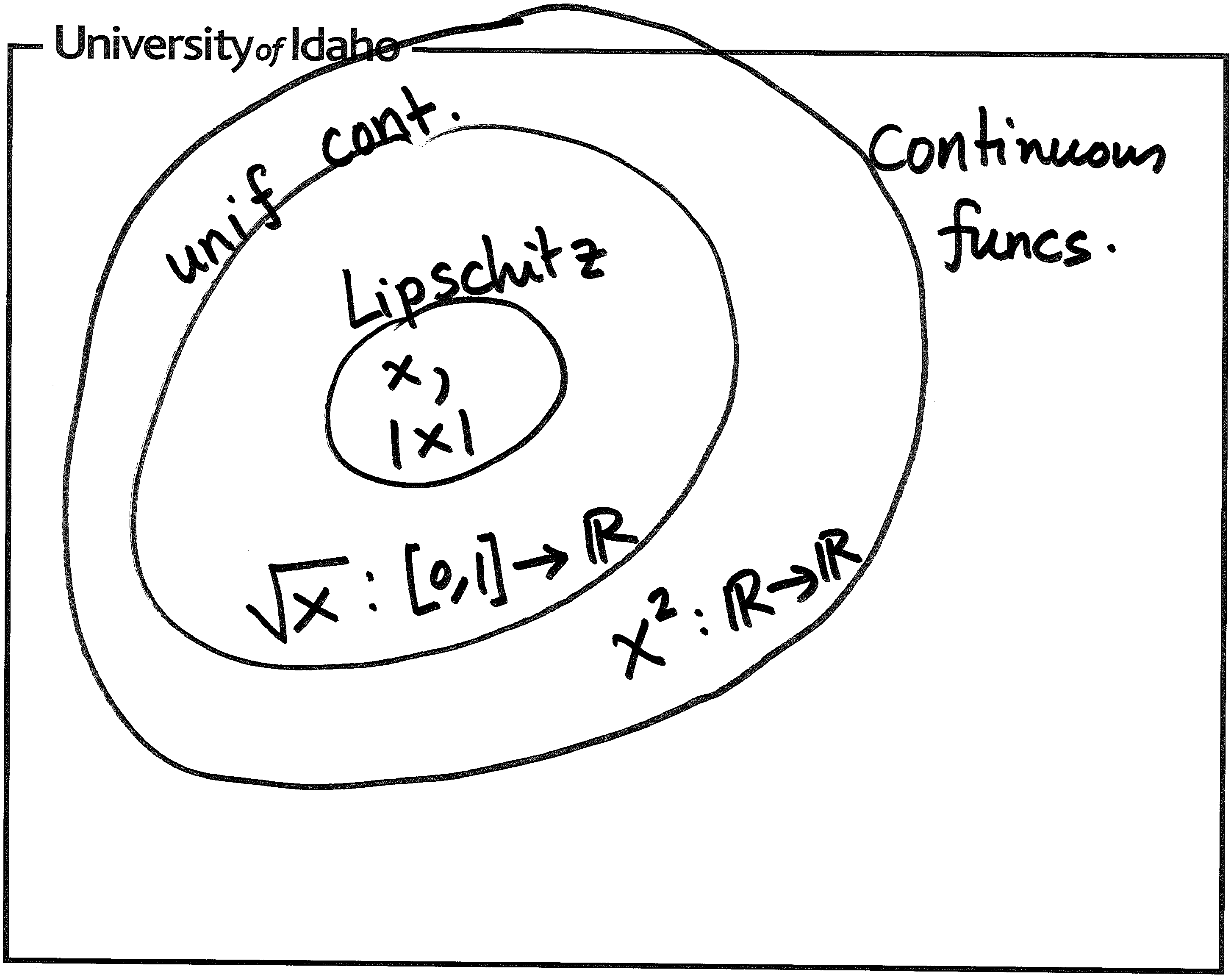

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$f: D \to \mathbb{R}$

f is unif. cont. if for any two seqs. $\{u_n\}$ and $\{v_n\}$ in D s.t. $|u_n - v_n| \to 0$

then

$$|f(u_n) - f(v_n)| \to 0.$$