MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 16

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$$f: D \longrightarrow \mathbb{R}$$

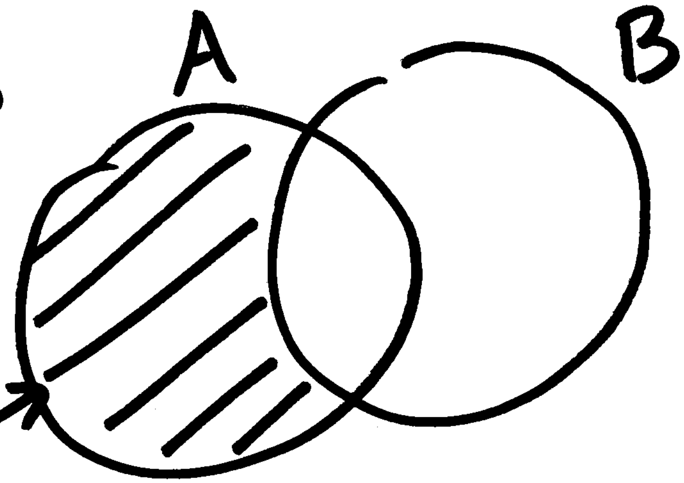
Limit of a function at a point x_0 that may not belong to D.

D - a set of reals

A, B : sets

$A \setminus B = \{x : x \in A \;\&\; x \notin B\}$

set minus



$A \setminus B$

$$D \setminus \{x_0\} = \{x \in D : x \neq x_0\}$$

Def [limit point of a set]:

A number x_0 is a <u>limit point</u> of D if there is a sequence in $D \setminus \{x_0\}$ that converges to x_0.

Examples

$\boxed{1}$ $D = [a, b]$

$a = ?$ Yes, a is a limit point. $a \in D$

$\left\{a + \frac{b-a}{n}\right\}_{n=1}^{\infty}$ is a

sequence in $D \setminus \{a\}$

that converges to a.

5 $D = [a, b]$

$b = ?$ Yes, b is a limit point $b \in D$.

$$\left\{ b - \frac{(b-a)}{n} \right\}_{n=1}^{\infty}$$

is a sequence in $D \setminus \{b\}$ that converges to b.

6

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$\boxed{2}$ $D = (a, b)$

$$\left\{ a + \frac{b-a}{2n} \right\}_{n=1}^{\infty} \in D \setminus \{a\}$$

$$\downarrow$$

$$a$$

a is a limit point of D

$a \notin D$.

7

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$D = (a, b)$

$b = ?$

$$\left\{ b - \frac{(b-a)}{2n} \right\}_{n=1}^{\infty} \in D \setminus \{b\}$$

$\downarrow$

b

b is a limit point of D

$b \notin (a, b)$.

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$(2, 5)$

a b

3 is a limit point

$2 \leq x_0 \leq 5$

x_0 : a limit point

$$\left\{ 3 + \frac{(5-3)}{2n} \right\} \to 3$$

a

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$\boxed{3}$

$D = (2, 10) \cup \{13\}$

13 - is not a limit point

There does not exist any sequence in $\underbrace{D \setminus \{13\}}_{= (2,10)}$ that converges to 13.

Def (limit of a function):

Given a function $f: D \to \mathbb{R}$ and a limit point $x_0 \in D$, we say $\lim_{x \to x_0} f(x) = \ell$ if whenever $\{x_n\} \to x_0$ & $\{x_n\} \in D \setminus \{x_0\}$ we have $\{f(x_n)\} \to \ell$

11

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Examples

1. Show that $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2.$

$$f(x) = \frac{x^2-1}{x-1}, \quad x_0 = 1,$$

Let $\{x_n\}$ be a sequence that converges to 1, $x_n \neq 1$

$$f(x_n) = \frac{x_n^2-1}{x_n-1} = \frac{(x_n-1)(x_n+1)}{x_n-1}$$

$$= x_n+1$$

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$$\{f(x_n)\} = \{x_n + 1\} \to 2$$

$$\Rightarrow \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

13

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Example

$\boxed{2}$ Find $\lim\limits_{x \to 0} \frac{1}{x}$.

$x_0 = 0$ Pick the sequence in $\mathbb{R} \setminus \{0\}$ given by $\{\frac{1}{n}\}$

$x_n = \frac{1}{n}$; $f(x_n) = \frac{1}{1/n} = n$

$\{f(x_n)\} = \{n\}$ does not converge because it is not bounded.

$\Rightarrow \lim\limits_{x \to 0} 1/x$ does NOT exist.