MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 17

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$$f : D \longrightarrow \mathbb{R}$$

Limit at a point

vs.

Continuity at a point

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Limit (Continuity)

$f : D \longrightarrow \mathbb{R}$, x_0 : limit point of D
(x_0 must be in D)

$$\lim_{x \to x_0} f(x) = \ell \quad \text{if}$$

(f is continuous at x_0)

whenever $\{x_n\} \to x_0$, $\{x_n\} \in D \setminus \{x_0\}$
($\{x_n\} \in D$)

we have $\{f(x_n)\} \longrightarrow \ell$.
($\{f(x_n)\} \longrightarrow f(x_0)$)

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Remark:

If x_0 is a limit point of D and $x_0 \in D$ then a function f is continuous at x_0 if and only if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

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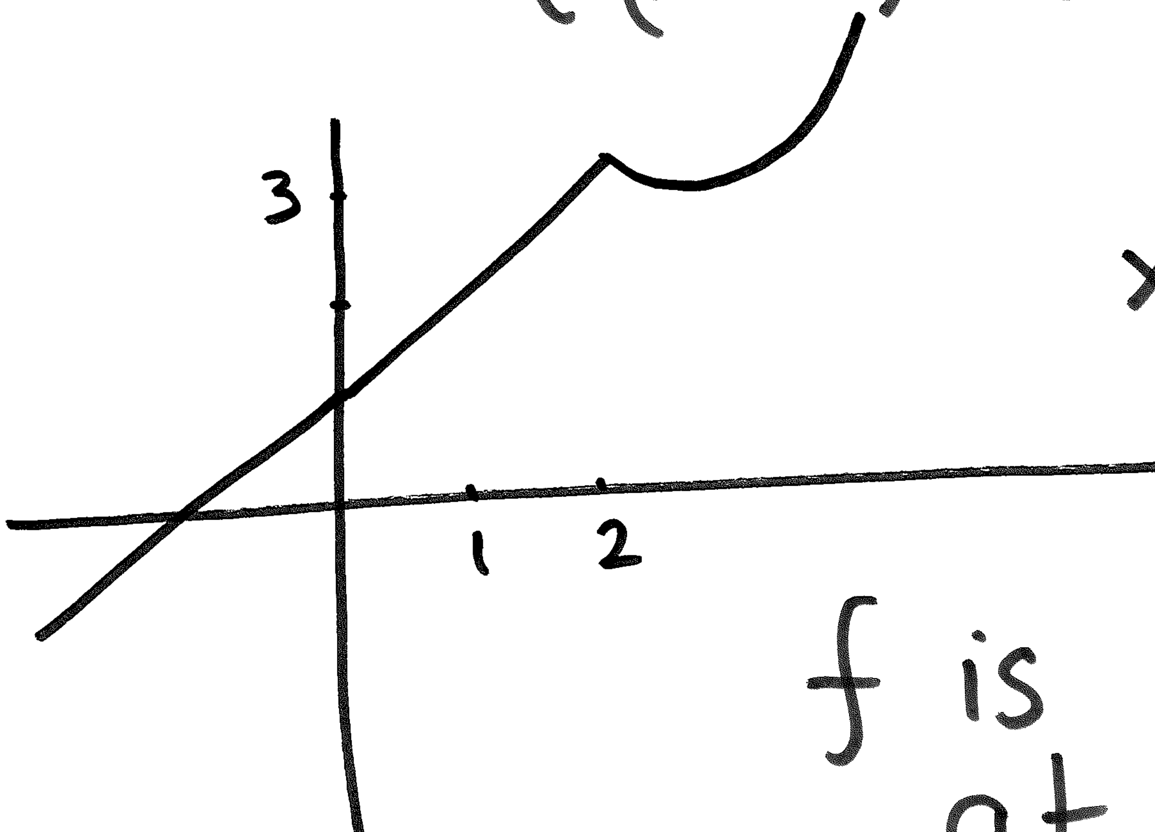
Examples

$\boxed{1}$ $f(x) = \begin{cases} x+1 & x \le 2 \\ (x-2)^2+3 & x > 2 \end{cases}$



$\lim_{x \to 2} f(x) = 3$

$f(2) = 3$

f is continuous at $x = 2$.

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$\boxed{2}$

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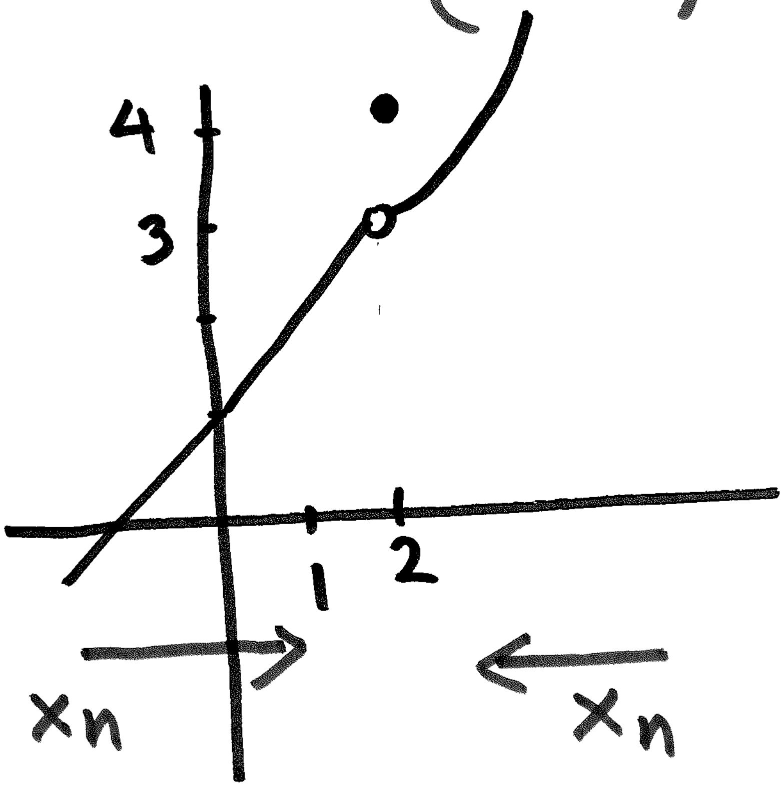
empty dot: point not on the graph

$$f(x) = \begin{cases} x+1 & x<2 \\ 4 & x=2 \\ (x-2)^2+3 & x>2 \end{cases}$$



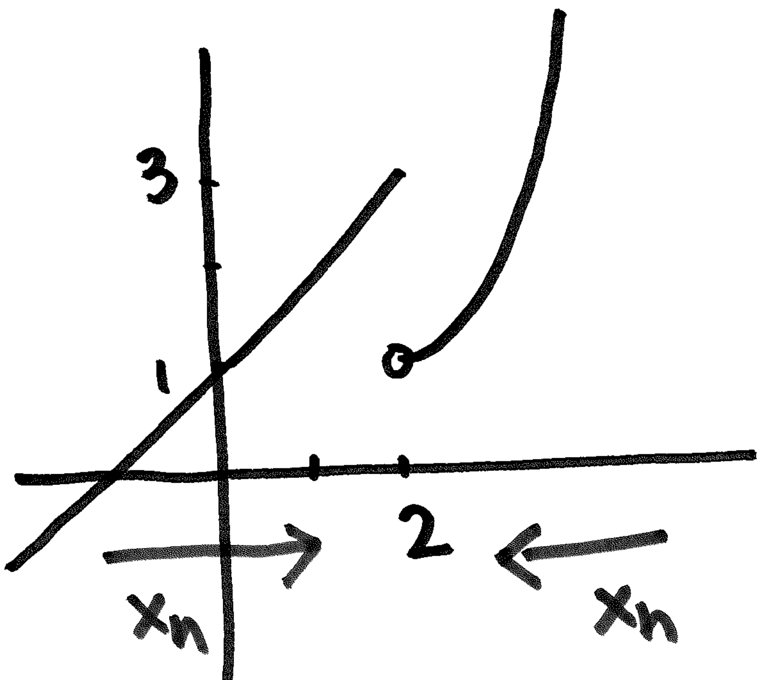
$$\lim_{x \to 2} f(x) = 3$$

$$f(2) = 4$$

$3 \neq 4 \Rightarrow f$ is not continuous at $x = 2$.

$\boxed{3}$

$$f(x) = \begin{cases} x+1 & x \le 2 \\ (x-2)^2+1 & x > 2 \end{cases}$$



$f(x_n) \to 3$ $\quad$ $f(x_n) \to 1$

$\lim_{x \to 2} f(x) =$ does not exist

$f(2) = 3 \neq \lim_{x \to 2} f(x)$

f is not continuous at $x = 2$

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$\boxed{4}$ $f(x) = \dfrac{\sin x}{x}$; $D = \mathbb{R} \setminus \{0\}$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

f is not continuous at $x = 0$ since $f(0)$ is not defined.

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Let

$$g(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0 \\ \cancel{0}\ 1 & x = 0 \end{cases}$$

$$\left.\begin{array}{l} \lim_{x \to 0} g(x) = 1 \\ g(0) = \cancel{0}\ 1 \end{array}\right\} \text{~~not cont.~~ continuous}$$

Properties of limits

<u>Thm</u>: For functions $f, g : D \to \mathbb{R}$ and a limit point x_0 of D, let

$$\lim_{x \to x_0} f(x) = A, \quad \lim_{x \to x_0} g(x) = B$$

Then

(a) $\lim_{x \to x_0} [f(x) \pm g(x)] = A \pm B$

(b) $\lim_{x \to x_0} f(x)\,g(x) = AB$

(c) If $B \neq 0$ and $g(x) \neq 0$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = A/B.$$