MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 18

- ε - δ criterion for continuity

- Differentiation

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University of Idaho $f : D \longrightarrow \mathbb{R}$ ε-δ definition

f is said to be continuous at $x_0 \in D$ if for any <u>given</u> $\varepsilon > 0$, there exist a $\delta > 0$ s.t.

find, depends on ε

$|f(x) - f(x_0)| < \varepsilon$ whenever

$|x - x_0| < \delta$.

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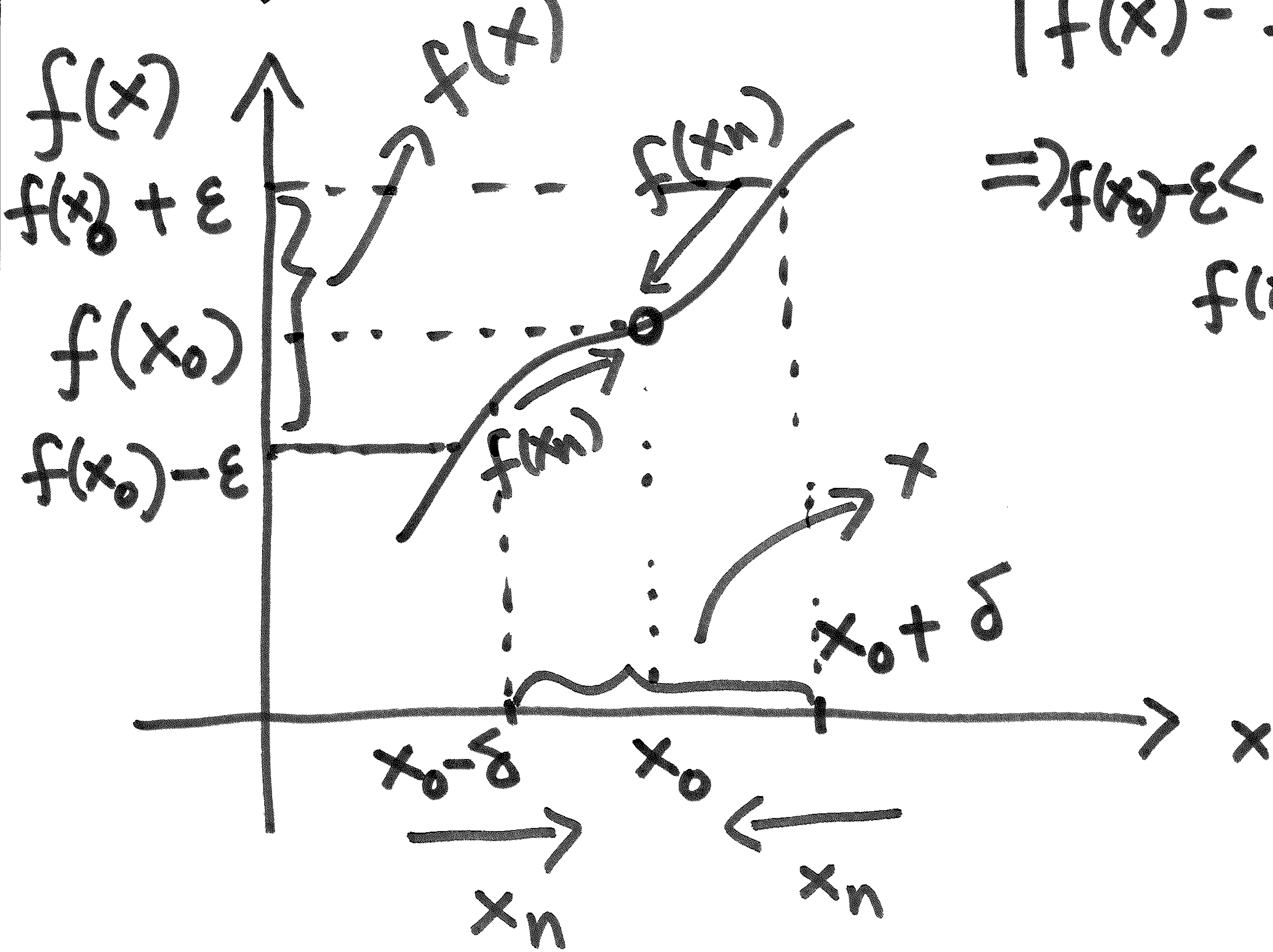
$$|f(x) - f(x_0)| < \varepsilon \Rightarrow f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$$



$$|x - x_0| < \delta \Rightarrow x_0 - \delta < x < x_0 + \delta$$

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Example

$f(x) = x^2$; $f : \mathbb{R} \longrightarrow \mathbb{R}$

Using ε-δ defn. show that f is continuous at $x_0 = 2$.

Given $\varepsilon > 0$, need to find δ s.t.

$|f(x) - f(2)| < \varepsilon$ when $|x-2| < \delta$.

4 $\quad f(x) = x^2$

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$$|x^2 - 4| < \varepsilon \Rightarrow \underbrace{|x+2|}_{\text{estimate}}\,\underbrace{|x-2|}_{<\delta} < \varepsilon$$

Let $\delta < 1$ *

$$|x-2| < 1 \Rightarrow -1 < x-2 < 1$$

$$\Rightarrow 1 < x < 3$$

$$\Rightarrow 3 < x+2 < 5$$

$$\Rightarrow 3 < |x+2| < 5$$

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$|x+2|\,|x-2|$

$< 5\delta$

Pick δ such that

$5\delta < \varepsilon$

$\Rightarrow \delta < \varepsilon/5$ **

Using * & **

$\delta < \min(1, \varepsilon/5)$ gives

$|f(x) - f(2)| < \varepsilon$ when

$|x-2| < \delta$.

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Differentiation

$f : D \longrightarrow \mathbb{R}$

$f = mx + b$ — Eq. of a st. line

m — slope ; $b = f(0)$

f is determined everywhere by knowing m & $f(0)$

m is fixed throughout the line

slope

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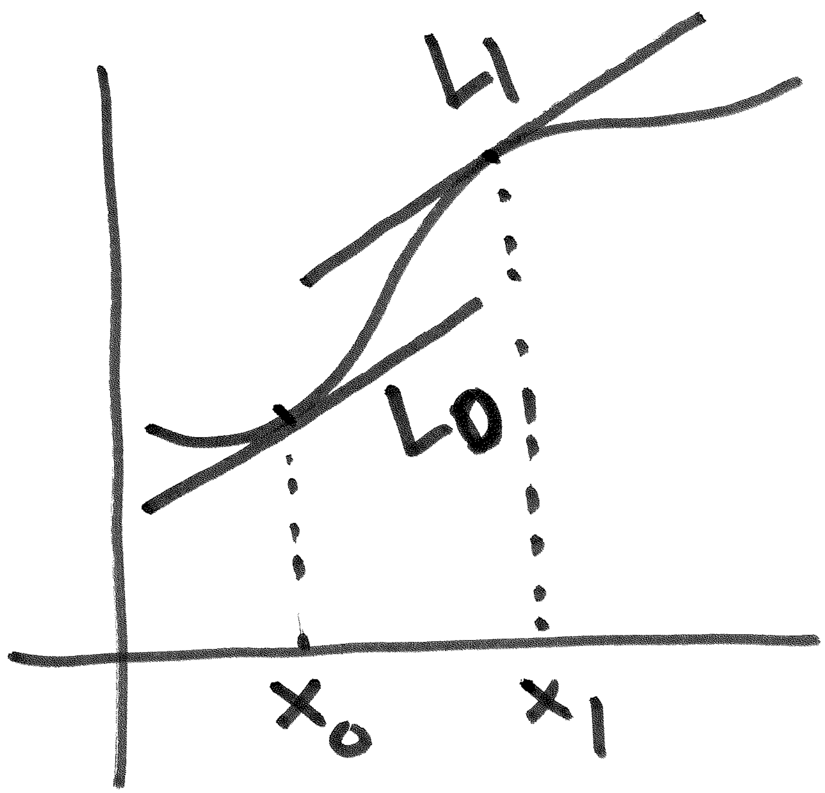
f – is not a straight line



L_0: tangent at x_0

m_0: slope of L_0

L_1: tangent at x_1

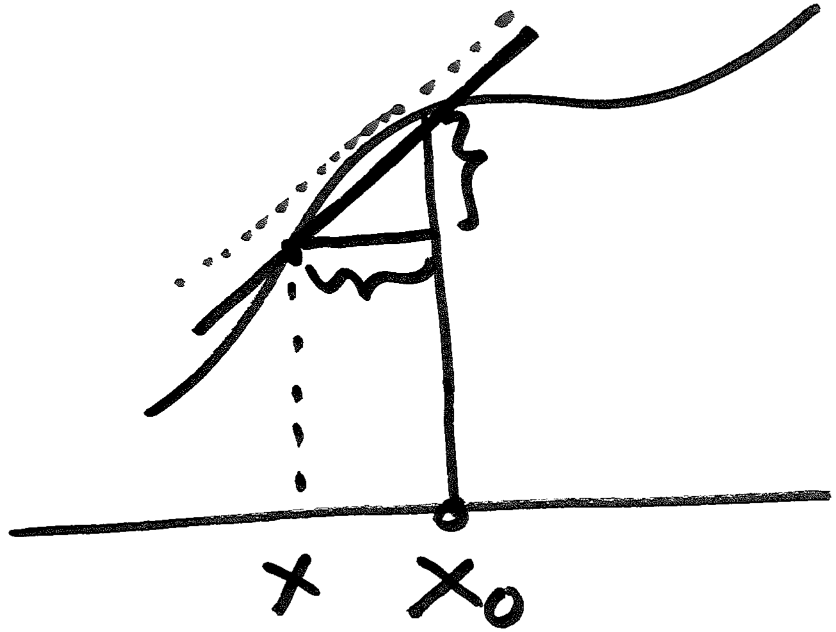
m_1: Slope of L_1

Study the slope ~~at~~ of the tangent at each point

The derivative of f at $x = x_0$ is defined by

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

provided this limit exists.



$$m_x = \frac{f(x) - f(x_0)}{x - x_0}$$

= slope of ℓ

As x approaches x_0 m_x approaches the slope of the tangent at x_0.

If the limit exists then

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

is called the derivative of f at x_0 and f is differentiable at x_0.

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1) $f(x) = x^{1/3}$; does not have a derivative at 0.

$$\lim_{x \to 0} \frac{\overbrace{x^{1/3}}^{f(x)} - \overbrace{0}^{f(0)}}{x - 0}$$

$$= \lim_{x \to 0} \frac{x^{1/3}}{x} = \lim_{x \to 0} \frac{1}{x^{2/3}}$$

which does not exist

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2) $f(x) = x^2$ is differentiable ~~for~~ at any real $x = a$.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} \frac{\cancel{(x-a)}(x+a)}{\cancel{x-a}}$$

$$= 2a$$