MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 19

University of Idaho Practice problems

#2 Given $\{a_n\} \to 0$, $\{b_n\}$ is bounded

Show that $\{a_n b_n\} \to 0$.

Proof: Given $\varepsilon > 0$, find $N \in \mathbb{N}$

s.t. $|a_n b_n - 0| < \varepsilon$, $n \geq N$

or, $|a_n b_n| < \varepsilon$, $n \geq \boxed{N}$

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$\exists M$ s.t. $|b_n| < M, \forall n \in \mathbb{N}$

Enough to find N s.t.

$|a_n||b_n| \leq \underbrace{|a_n| M < \varepsilon}$, $n \geq N$.

or, $|a_n| < \varepsilon/M$, $n \geq N$

$|a_n - 0| < \varepsilon/M$, $n \geq N$

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Since $\{a_n\} \to 0$. Given

$\varepsilon_1 = \varepsilon/M$, $\exists N_1$ s.t.

$|a_n - 0| < \varepsilon/M$, $n \geq N_1$.

Take $N = N_1$

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Proving convergence/

- Definition
- MCT
- Squeeze/ Sandwich Thm

Divergence

A seq converges to a $\Longleftrightarrow$ every subsequence converges to a

used for proving diverg-ence

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After #6

Note : For 5. & 6. assume that polynomials are continuous

#6 $f: \mathbb{R} \to \mathbb{R}$ is <u>continuous</u>. Show that $g(x) = f(\underbrace{x^2 + x^3})$. Show that g is continuous.

$h(x) = x^2 + x^3$

$g(x) = f \circ h(x)$

h is cont. because it is a polynomial

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g is continuous since it is the composition of two continuous functions.

Proving uniform continuity

- Definition
- cont. funcs. <u>on</u> closed & bounded intervals are unif. cont.
- Lipschitz functions are unif. cont.

$\{u_n\}, \{v_n\}$ s.t. $[u_n - v_n] \to 0$

Then show $f(u_n) - f(v_n) \to 0$.

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#9 Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ is unif. cont.

We show that f is Lipschitz, i.e., $\exists\, L$ s.t

$$|f(x) - f(y)| < L\,|x-y| \quad \forall\, x, y \in \mathbb{R}.$$

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$|5| < 10 \quad \begin{cases} -5 < 10 \\ 5 < 10 \end{cases}$

$$|x| = |x - y + y|$$

Triangle

$$\leq |x - y| + |y|$$

$$\Rightarrow \underbrace{|x| - |y|}_{a} \leq |x - y|$$

$$|y| = |y - x + x|$$

$$\leq |y - x| + |x|$$

$$\overbrace{|y| - |x|}^{-a} \leq |x - y|$$

$$\Rightarrow \big|\,|x| - |y|\,\big| \leq |x - y|$$

$$|f(x) - f(y)| = \big||x| - |y|\big| \leq |x-y|$$

$\Rightarrow$ f is Lipschitz with Lipschitz const. $L = 1$.

$\Rightarrow$ f is unif. cont.

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$f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$

11 (c) $\lim_{x \to 0} \dfrac{1 + \frac{1}{x^2}}{1 + \frac{1}{x}}$. $f(x) = \dfrac{1 + \frac{1}{x^2}}{1 + \frac{1}{x}}$

Def: For any seq $\{x_n\} \in \mathbb{R} \setminus \{0\}$

$\{f(x_n)\} \to l$.

Let $\{x_n = \frac{1}{n}\}$. $f(x_n) = \dfrac{1+n^2}{1+n}$

Show that $\{f(x_n)\}$ diverges.

$\{x_n = \frac{1}{n}\} \longrightarrow 0$

$\in \mathbb{R} \setminus \{0\}$

$\left\{ \frac{1+n^2}{1+n} \right\}$ is unbounded

$\Rightarrow$ it diverges.

$\Rightarrow$ the limit does not exist.