MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 20

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Recall:

$f : D \longrightarrow \mathbb{R}$

The derivative of f at x_0 is defined by

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

if it exists and is denoted by $f'(x_0)$.

We say that f is differentiable at x_0.

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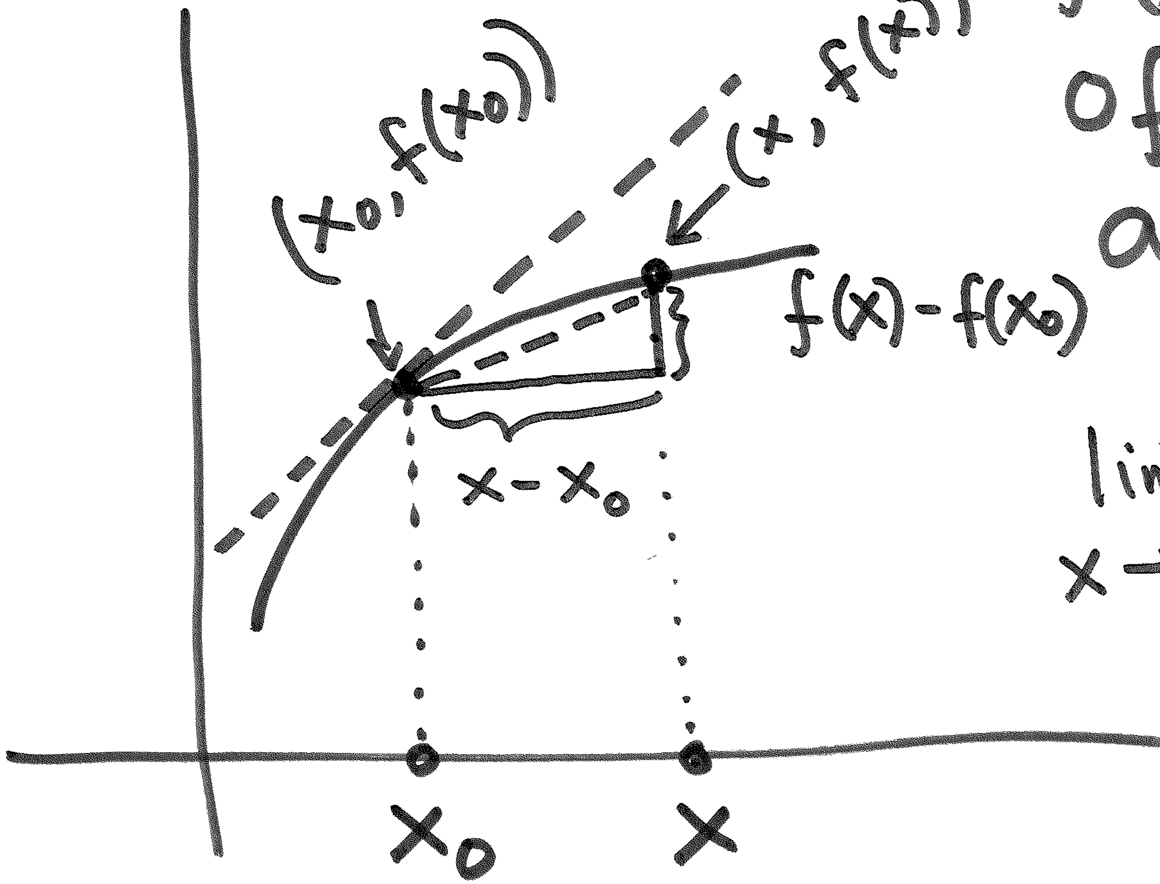

$f'(x_0)$ = slope of the tangent at x_0

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$f(x) = x^2$ is differentiable at any $x_0 \in \mathbb{R}$

$f(x) = x^{1/3}$ is not differentiable at $x_0 = 0$.

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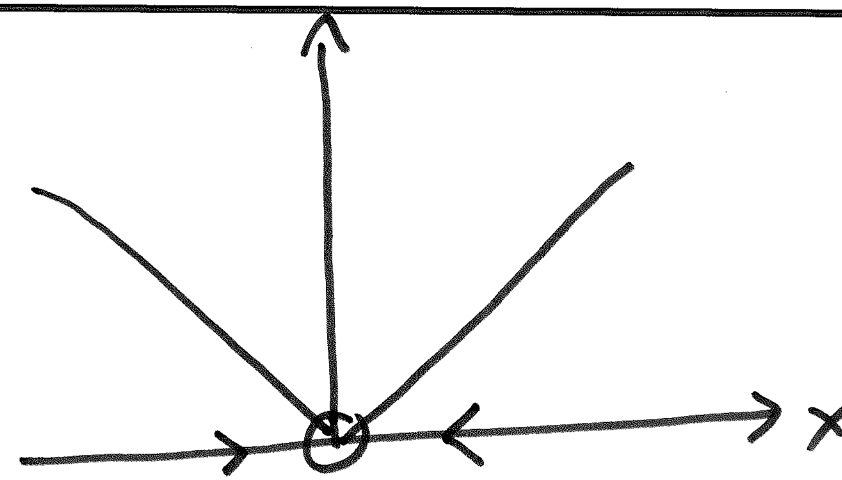
$f(x) = |x|$



Let $x_0 = 0$.

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{|x|}{x} \ \text{does not exist}$$

$$\lim_{\substack{x \to 0 \\ x > 0}} \frac{x}{x} = 1 \ ; \ \lim_{\substack{x \to 0 \\ x < 0}} \frac{-x}{x} = -1$$

$$1 \neq -1$$

$\Rightarrow f$ is not differentiable at $x_0 = 0$.

$D = [a, b]$ $\qquad x_0 = a$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$f(x)$ is undefined for $x < a$.

$D = (a, b)$

$x_0 \in D$.

Def: An interval (a, b) containing x_0 is called a neighborhood of x_0.

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Theorem

If $f : D \to \mathbb{R}$ is differentiable at $x = x_0$ (where D is some neighborhood of x_0) then f is Continuous at x_0.

Proof: We need to show

$$\lim_{x \to x_0} f(x) = f(x_0).$$

$$\Rightarrow \lim_{x \to x_0} \left[f(x) - f(x_0) \right] = 0.$$

$$\lim_{x \to x_0} [f(x) - f(x_0)] =$$

$$\lim_{x \to x_0} [f(x) - f(x_0)] \frac{(x - x_0)}{(x - x_0)}$$

$$= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \right] \lim_{x \to x_0} (x - x_0)$$

$$= f'(x_0) \cdot 0 = 0$$

$\square$

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Properties of differentiable funcs.

Theorem Let $f, g : D \longrightarrow \mathbb{R}$ be differentiable at $x_0 \in D$ (where D is a neighborhood of x_0). Then

(i) $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$

Product rule

(ii) $(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$

(iii) If $g(x) \neq 0, \forall x \in D$, $\left(\frac{1}{g}\right)'(x_0) = \frac{-g'(x_0)}{(g(x_0))^2}$

(iv) $\left(\frac{f}{g}\right)'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}$ → Quotient rule

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Proof of (ii)

$$(fg)'(x_0) = \lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{(fg)(x) - f(x_0)g(x) + f(x_0)g(x) - (fg)(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0}$$

$f(x_0)$

$$= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0}\right] g(x) + \lim_{x \to x_0} \left[\frac{g(x) - g(x_0)}{x - x_0}\right] f(x)$$

$$= f'(x_0)\, g(x_0) + g'(x_0)\, f(x_0)$$

Proof of (iii)

$$\left(\frac{1}{g}\right)'(x_0) = \lim_{x \to x_0} \frac{\frac{1}{g}(x) - \frac{1}{g}(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{1}{g(x)g(x_0)} \frac{g(x_0) - g(x)}{x - x_0}$$

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$$= \frac{1}{g(x_0)^2}\left(-g'(x_0)\right)$$

$\square$